5.3.5 Let the functions f_n be defined on $[0,1]$ by

$$f_n(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 1 - n\left(x - \frac{1}{2}\right) & \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

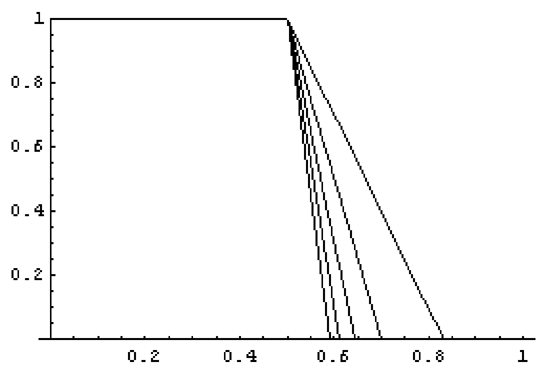
and define

$$f(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1 \end{cases}.$$

5.3.5.a Prove that $f_n \to f$ pointwise on $[0,1]$. Hint: draw the graph of f_n.

Proof of exercise 5.3.5.a:

The graphs of f_n for $n \in \{3, 5, 7, 9, 11\}$ are:



$f_n \to f$ pointwise on $[0,1]$ if $\forall \varepsilon > 0, \forall x \in [0,1], \exists N(\varepsilon, x) > 0 \ni \forall n \ge N$, $|f_n(x) - f(x)| < \varepsilon$. Take $\varepsilon > 0$ arbitrary.

$\forall x \in \left[0, \frac{1}{2}\right]$, $f_n(x) = 1 = f(x)\ \forall n$. Thus, $|f_n(x) - f(x)| = |1-1| = 0 < \varepsilon$.

Take $x \in \left(\frac{1}{2}, 1\right]$ arbitrary. Pick $N = \frac{2}{2x+1} + 1$. Then $\forall n \ge N$, $f_n(x) = 0 = f(x)$. Thus, $\forall n \ge N$, $|f_n(x) - f(x)| = |0-0| = 0 < \varepsilon$. Therefore, $f_n \to f$ pointwise on $[0,1]$. Q.E.D.

5.3.5.b Prove that $\|f - f_n\|_\infty = 1$ for each n so that f_n does not converge to f in the sup norm.

Proof of exercise 5.3.5.b:

$\|f - f_n\|_\infty = \sup_{x \in [0,1]} |f(x) - f_n(x)|$. Take n arbitrary. Since f_n is continuous on $[0,1]$, it is continuous at $x = \frac{1}{2} \Rightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ni \forall x \in \left[\frac{1}{2}, \frac{1}{2} + \delta\right]$, we have $\left| f_n(x) - f_n\left(\frac{1}{2}\right) \right| = \left| f_n(x) - 1 \right| < \varepsilon.$

$\Rightarrow 1 - f_n(x) = |1| - |f_n(x)| \le |f_n(x) - 1| < \varepsilon$ by exercise 1.1.10.

$\Rightarrow f_n(x) > 1 - \varepsilon.$ Since $\forall x \in \left(\frac{1}{2}, 1\right],\ f(x) = 0,$ it follows that, choosing

$x_0 \in \left[\frac{1}{2}, \frac{1}{2} + \delta\right]$, we have: $\|f - f_n\|_\infty \ge |f(x_0) - f_n(x_0)| = f_n(x_0) > 1 - \varepsilon.$

Since $\forall x \in [0,1],$ we have $\|f - f_n\|_\infty \le 1$, it follows that $1 - \varepsilon < \|f - f_n\|_\infty \le 1$.

Since this holds for all $\varepsilon > 0,$ it must be the case that $\|f - f_n\|_\infty = 1.$

Since n was arbitrary, this holds $\forall n \in \mathbf{N}$. Q.E.D.

5.3.5.c Explain how you could have predicted the result of part (b) simply by using Theorem 5.2.1.

Answer to exercise 5.3.5.c:

Since $\forall n,\ f_n$ is continuous on $[0,1],$ theorem 5.2.1 would have suggested that, had $f_n \to f$ uniformly on $[0,1],$ then f would have been continuous on $[0,1].$ Since f is not continuous on $[0,1],$ it then follows that f_n does not converge to f uniformly on $[0,1].$ Thus, by theorem 5.3.2, f_n does not converge to f in the sup norm.

5.3.5.d Prove that $\|f - f_n\|_1 \to 0$ as $n \to \infty.$

Since $\forall x \in [0,1],\ f(x) \ge 0,$ it follows that $|f(x)| = f(x).$

$$\|f - f_n\|_1 = \int_0^1 |f(x) - f_n(x)|dx = \int_0^1 [f(x) - f_n(x)]dx = \int_0^1 f(x)dx - \int_0^1 f_n(x)dx$$

$$= \int_0^{\frac{1}{2}} 1dx + \int_{\frac{1}{2}}^1 0dx - \int_0^{\frac{1}{2}} 1dx - \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left[1 - n\left(x - \frac{1}{2}\right)\right]dx - \int_{\frac{1}{2}+\frac{1}{n}}^1 0dx$$

$$= \frac{1}{2} + 0 - \frac{1}{2} - \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left[1 - n\left(x - \frac{1}{2}\right)\right]dx - 0 = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \left[n\left(x - \frac{1}{2}\right) - 1\right]dx$$

$$= \left[\frac{nx^2}{2} - \frac{nx}{2} - x\right]_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} = \left[\frac{n\left(\frac{1}{2} + \frac{1}{n}\right)^2}{2} - \frac{n\left(\frac{1}{2} + \frac{1}{n}\right)}{2} - \left(\frac{1}{2} + \frac{1}{n}\right)\right]$$

$$- \left[\frac{n\left(\frac{1}{2}\right)^2}{2} - \frac{n\left(\frac{1}{2}\right)}{2} - \left(\frac{1}{2}\right)\right] = \frac{n}{8} + \frac{1}{2} + \frac{1}{2n} - \frac{n}{4} - \frac{1}{2} - \frac{1}{2} - \frac{1}{n} - \frac{n}{8} + \frac{n}{4} + \frac{1}{2}$$

$$= \frac{1}{2n} - \frac{1}{n}$$

Where the fundamental theorem of calculus, part I and theorem 3.3.7 were used several times.

Thus, $\|f - f_n\|_1 = \frac{1}{2n} - \frac{1}{n} \Rightarrow \lim_{n\to\infty}\|f - f_n\|_1 = \lim_{n\to\infty}\left(\frac{1}{2n} - \frac{1}{n}\right) = 0.$ Q.E.D.