5.3.4 For $f \in C[a,b]$, define $\|f\|_1 = \int_a^b |f(x)|dx$. Show that $\|\cdot\|_1$ satisfies the three properties, (a), (b), and (c), of Proposition 5.3.1.

5.3.4.a $\|f\|_1 > 0$ if f is not identically 0 on $[a,b]$ and $\|f\|_1 = 0$ when $f \equiv 0$.

Proof of exercise 5.3.4.a:

Suppose $f \equiv 0$. Then $\forall x \in [a,b],\ f(x) = 0 \Rightarrow \forall x \in [a,b], |f(x)| = 0.$

Thus, $\|f\|_1 = \int_a^b 0dx = 0.$

Suppose f is not identically zero. Since $f \in C[a,b]$, it follows that $|f| \in C[a,b]$ by lemma 5.2.5.1. Then $\exists\, c \in [a,b] \ni |f(c)| = d > 0$. Pick $\varepsilon = \frac{d}{2}$. Since $|f|$ is continuous on $[a,b]$, $\exists\, \delta\left(\frac{d}{2}\right) > 0 \ni \forall\ x \in [c-\delta, c+\delta], |f(x) - f(c)| < \frac{d}{2}$

$\Rightarrow -\frac{d}{2} < f(x) - d < \frac{d}{2} \Rightarrow \frac{d}{2} < f(x) < \frac{3d}{2}$. That is, $|f(x)| > \frac{d}{2}$.

Thus, $\|f\|_1 = \int_a^b |f(x)|dx \geq \int_{c-\delta}^{c+\delta} |f(x)|dx > \frac{d}{2}(2\delta) > 0$. Q.E.D.

5.3.4.b For every $\alpha \in \mathbf{R}$, we have $\|\alpha f\|_1 = |\alpha|\,\|f\|_1$.

Proof of exercise 5.3.4.b:

By proposition 1.1.2.b, we have that $\forall x \in [a,b], |\alpha f(x)| = |\alpha||f(x)|$

$\Rightarrow |\alpha f(x)| \leq |\alpha||f(x)|$ and $|\alpha f(x)| \geq |\alpha||f(x)|\ \forall x \in [a,b].$

By theorem 3.3.4, we have that:

$\int_a^b |\alpha f(x)|dx \leq \int_a^b |\alpha||f(x)|dx$ and $\int_a^b |\alpha f(x)| \geq \int_a^b |\alpha||f(x)|.$

$\Rightarrow \int_a^b |\alpha f(x)|dx = \int_a^b |\alpha||f(x)|dx$

By theorem 3.3.3, we have:

$\int_a^b |\alpha||f(x)|dx = |\alpha|\int_a^b |f(x)|dx$

That is, $\|\alpha f\|_1 = \int_a^b |\alpha f(x)|dx = \int_a^b |\alpha||f(x)|dx = |\alpha|\int_a^b |f(x)|dx = |\alpha|\|f\|_1$. Q.E.D.

5.3.4.c $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ (the triangle inequality).

Proof of exercise 5.3.4.c:

By proposition 1.1.2.c, we have that $\forall x \in [a,b], |f(x) + g(x)| \leq |f(x)| + |g(x)|$.

Using theorem 3.3.4, we have:

$\int_a^b |f(x) + g(x)| \leq \int_a^b (|f(x)| + |g(x)|)dx$

By theorem 3.3.3, we have:

$\int_a^b (|f(x)| + |g(x)|)dx = \int_a^b |f(x)|dx + \int_a^b |g(x)|dx$

Combining these, we have:

$$\|f+g\|_1=\int_a^b|f(x)+g(x)|\leq\int_a^b\left(|f(x)|+|g(x)|\right)dx$$

$$=\int_a^b|f(x)|dx+\int_a^b|g(x)|dx=\|f\|_1+\|g\|_1.$$ Q.E.D.