5.3.2 Let f and g be continuous functions on $[a,b]$.

5.3.2.a Use the triangle inequality to prove that

$$\left| \|f\|_\infty - \|g\|_\infty \right| \le \|f-g\|_\infty .$$

We know that f and g are continuous functions on $[a,b]$. By theorem 3.2.1, both f and g are bounded. Thus, the notion of sup norm may be used.

Following the same line of logic as in exercise 1.1.10, we have:

$\|f\|_\infty = \|f+0\|_\infty = \|f-g+g\|_\infty \le \|f-g\|_\infty + \|g\|_\infty$ by the triangle inequality.

Thus, $\|f\|_\infty - \|g\|_\infty \le \|f-g\|_\infty$. Similarly, we have:

$\|g\|_\infty = \|g+0\|_\infty = \|g-f+f\|_\infty \le \|g-f\|_\infty + \|f\|_\infty$ by the triangle inequality.

Thus, $\|g\|_\infty - \|f\|_\infty \le \|g-f\|_\infty = \|f-g\|_\infty$.

Since $\left| \|f\|_\infty - \|g\|_\infty \right| = \begin{cases} \|f\|_\infty - \|g\|_\infty if \|f\|_\infty - \|g\|_\infty \ge 0 \\ \|g\|_\infty - \|f\|_\infty if \|f\|_\infty - \|g\|_\infty < 0 \end{cases}$, it follows that

$\left| \|f\|_\infty - \|g\|_\infty \right| \le \|f-g\|_\infty$. Q.E.D.

5.3.2.b Suppose that $f_n \to f$ in the sup norm. Prove that $\|f_n\|_\infty \to \|f\|_\infty$.

Proof of exercise 5.3.1.b:

We know that f and g are continuous functions on $[a,b]$. By theorem 3.2.1, both f and g are bounded. Thus, the notion of sup norm may be used.

Since $f_n \to f$ in the sup norm, $\forall \varepsilon > 0, \exists N(\varepsilon) > 0 \ni \forall n \ge N$, $\|f_n - f\|_\infty \le \varepsilon$.

By part a, we know that $\left| \|f_n\|_\infty - \|f\|_\infty \right| \le \|f_n - f\|_\infty$.

Choosing $n \ge N$, we have:

$\left| \|f_n\|_\infty - \|f\|_\infty \right| \le \|f_n - f\|_\infty \le \varepsilon$. That is, $\|f_n\|_\infty \to \|f\|_\infty$. Q.E.D.