5.3.1 Let f_n be a sequence of bounded functions on a set E, and suppose that f is a bounded function such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Prove that $\{f_n\}$ is a Cauchy sequence in the sup norm.

Proof of exercise 5.3.1:

Since $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, $\forall \varepsilon > 0, \exists N(\varepsilon) > 0 \ni \forall n \geq N$, $\|f_n - f\|_\infty \leq \frac{\varepsilon}{2}$.

Choose $n, m \geq N$. Then, $\|f_n - f_m\|_\infty \leq \|f_n - f\|_\infty + \|f - f_m\|_\infty \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.