5.2.6 Let $\{f_n\}$ be a sequence of continuous functions that converges uniformly on $[0,1]$.

Show that there is an M so that $|f_n(x)| \le M \ \forall n$ and $\forall x \in [0,1]$.

Proof of exercise 5.2.6:

For each n, f_n is continuous on $[0,1]$. By theorem 3.2.1, $\exists M_n > 0 \ni \forall x \in [0,1]$, $|f_n(x)| \le M_n$.

Since $f_n \xrightarrow[n\to\infty]{} f$ uniformly on $[0,1]$, by theorem 5.2.1, f is continuous on $[0,1]$.

Thus, by theorem 3.2.1, $\exists M^* > 0 \ni \forall x \in [0,1]$, $|f(x)| \le M^*$.

Since $f_n \xrightarrow[n\to\infty]{} f$ uniformly on $[0,1]$, we have that $\forall \varepsilon > 0, \exists N(\varepsilon) > 0 \ni \forall n \ge N$, $\forall x \in [0,1]$, $|f_n(x) - f(x)| \le \varepsilon$. Pick $\varepsilon < 1$.

Then $\forall n \ge N(\varepsilon)$, $|f_n(x)| - |f(x)| \le |f_n(x) - f(x)| \le \varepsilon$.

$\Rightarrow |f_n(x)| \le |f(x)| + \varepsilon < |f(x)| + 1 \le M^* + 1$.

Define $M_{\lim} \equiv M^* + 1$.

Then, $\forall n = 1,2,\ldots,N$, we have: $|f_n(x)| \le M_n \ \forall x \in [0,1]$.

$\forall n = N+1, N+2, \ldots$, we have: $|f_n(x)| \le M_{\lim} \ \forall x \in [0,1]$.

Choose $M = \max\{M_1, M_2, \ldots, M_N, M_{\lim}\}$.

$\forall n$, $|f_n(x)| \le M \ \forall x \in [0,1]$. Q.E.D.