

5.2.5 Suppose that f is continuous and $|f(x)| < 1$ for all $x \in [a, b]$. Prove that

$$\lim_{n \rightarrow \infty} \int_a^b (f(t))^n dt = 0.$$

Lemma 5.2.5.1:

If f is continuous on $[a, b]$, then $|f|$ is continuous on $[a, b]$.

Proof of lemma 5.2.5.1:

Suppose f is continuous on $[a, b]$. $\forall x_0 \in [a, b], \forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \ni$

$\forall x \in [a, b]$ satisfying, $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$. By exercise

1.1.10, $\forall f(x), f(x_0) \in \mathbf{R}, \|f(x) - f(x_0)\| \leq |f(x) - f(x_0)|$.

Pick $x \in [a, b]$ satisfying $|x - x_0| < \delta$. Then we have

$\|f(x) - f(x_0)\| \leq |f(x) - f(x_0)| \leq \varepsilon$. That is, $|f|$ is continuous at x_0 . Since x_0 was arbitrary on $[a, b]$, it follows that $|f|$ is continuous on $[a, b]$.

Lemma 5.2.5.2:

If f is continuous on $[a, b]$ and $\forall x \in [a, b]$, we have $|f(x)| < 1$, then

$$\sup_{x \in [a, b]} |f(x)| = M < 1.$$

Proof of lemma 5.2.5.2:

Let $M = \sup_{x \in [a, b]} |f(x)|$. Since $|f|$ is continuous (by lemma 5.2.5.1) on a closed

interval, by theorem 3.2.2, $\exists c \in [a, b] \ni |f(c)| = M$. Since $\forall x \in [a, b], |f(x)| < 1$, it follows that $M = |f(c)| < 1$.

Proof of exercise 5.2.5:

By lemma 5.2.5.2, we have that $\forall x \in [a, b], |f(x)| \leq M < 1$.

Define $f_n(x) \equiv |(f(x))^n|$. Choose $N = \left\lceil \frac{\ln \varepsilon}{\ln M} \right\rceil$. Then, $\forall x \in [a, b], \forall n \geq N$, we have:

$|f_n(x) - 0| = |(f(x))^n| = |f(x)|^n \leq M^n \leq M^{\frac{\ln \varepsilon}{\ln M}} = \varepsilon$. That is, f_n converges uniformly to 0 on $[a, b]$.

By theorem 5.2.2, it then follows that:

$$\lim_{n \rightarrow \infty} \int_a^b (f(t))^n dt = \lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b \lim_{n \rightarrow \infty} f_n(t) dt = \int_a^b 0 dt = 0.$$

Q.E.D.