5.2.4 Let $\{f_n\}$ be a sequence of continuous functions on a finite interval $[a,b]$ that converges uniformly to f. Show that for all continuous functions g on $[a,b]$,

$$\lim_{n\to\infty}\int_a^b f_n(t)g(t)dt = \int_a^b f(t)g(t)dt.$$

Proof of exercise 5.2.4:

The result follows from theorem 5.2.2 if we can show that $f_n(x)g(x) \to f(x)g(x)$ uniformly.

Since $f_n(x) \to f(x)$ uniformly, we have that $\forall \varepsilon > 0, \ \exists N(\varepsilon) > 0 \ni$

$\forall x \in [a,b], \forall n \geq N, \left| f_n(x) - f(x) \right| \leq \dfrac{\varepsilon}{M}.$

Since $g(x)$ is continuous on $[a,b]$, by Theorem 3.2.1, $\exists M > 0 \ni \forall x \in [a,b]$, $\left|g(x)\right| \leq M$.

Thus, $\forall n \geq N$, we have:

$\left| f_n(x)g(x) - f(x)g(x) \right| = \left| (f_n(x) - f(x))g(x) \right| = \left|g(x)\right| \left| f_n(x) - f(x) \right|$

$\leq M \left| f_n(x) - f(x) \right| \leq M \dfrac{\varepsilon}{M} = \varepsilon.$

Thus, $f_n(x)g(x) \to f(x)g(x)$ uniformly on $[a,b]$.

Since $f_n(x)$ is continuous $\forall n$, and since $g(x)$ is continuous, $f_n(x)g(x)$ is continuous $\forall n$ by theorem 3.1.1.c.

By theorem 5.2.2, since $\{f_n g\}$ is a sequence of continuous functions on $[a,b]$, and $f_n(x)g(x) \to f(x)g(x)$ uniformly as $n \to \infty$, we have that:

$$\lim_{n\to\infty}\int_a^b f_n(t)g(t)dt = \int_a^b \lim_{n\to\infty} f_n(t)g(t)dt = \int_a^b f(t)g(t)dt. \quad \text{Q.E.D.}$$