5.2.11 Let *f* be a continuous function on $[0,\infty)$ which equals zero outside the interval $[a,b]$. For each $\lambda > 0$, define

$$F(\lambda) \equiv \int_0^\infty e^{-\lambda x} f(x)dx$$

By using Theorem 5.2.4, prove that *F* is infinitely often continuously differentiable on $(0,\infty)$. Remark: *F* is called the Laplace transform of *f*.

Proof of exercise 5.2.11:

$$F(\lambda) \equiv \int_0^\infty e^{-\lambda x} f(x)dx = \int_a^b e^{-\lambda x} f(x)dx.$$

By theorem 5.2.4, we have: $\dfrac{dF(\lambda)}{d\lambda} = \int_a^b \dfrac{d(e^{-\lambda x} f(x))}{d\lambda} dx = \int_a^b - xe^{-\lambda x} f(x)dx$

which is a continuous function of λ.

Proceeding iteratively:

$$\frac{d^2F(\lambda)}{d\lambda^2} = \int_a^b (-x)^2 e^{-\lambda x} f(x)dx$$

$$\vdots$$

$$\frac{d^nF(\lambda)}{d\lambda^n} = \int_a^b (-x)^n e^{-\lambda x} f(x)dx$$

$$\vdots$$

For each *n*, $\dfrac{d^nF(\lambda)}{d\lambda^n}$ is a continuous function of λ. Therefore, we have that $F \in C^\infty(0,\infty)$. Q.E.D.