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Problem Set 3

5.2.1 Let $f_n(x) = e^{-nx}$ on the interval $[0,1]$. Explain why the sequence of functions $\{f_n\}$ converges pointwise on $[0,1]$. What is the limiting function? Is it continuous? Is the convergence uniform?

Answer to exercise 5.2.1:

Case I: $x = 0$.

$f_n(0) = e^{-n(0)} = e^0 = 1 \forall n \in \mathbf{N}$. Thus, f_n converges pointwise on 0.

Case II: $x \in (0,1]$.

Take $N(\varepsilon, x) = \left\lceil \frac{-\ln \varepsilon}{x} \right\rceil$. Then, $\forall n \geq N(\varepsilon, x)$,

$f_n(x) = e^{-nx} \leq e^{-\left(\frac{-\ln \varepsilon}{x}\right)x} = e^{\ln \varepsilon} = \varepsilon$. That is, f_n converges pointwise on $(0,1]$.

Thus, f_n converges pointwise on $[0,1]$.

f_n converges pointwise to $f(x) = \begin{cases} 0, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$. This function is clearly not continuous as it has a jump discontinuity at $x = 0$.

Suppose the convergence were uniform. Then the limiting function $f(x)$ would be continuous. Since $f(x)$ is not continuous, we can conclude that the convergence was not uniform.

Another way to see that the convergence is not uniform is to pick $\varepsilon > \frac{1}{e}$. Then,

$\forall n, \exists x \in [0,1] \ni |f_n(x)| > \varepsilon$. In particular, set $x = \frac{1}{n}$.