5.1.9 Suppose that g is a continuous function on $[0,1]$ and that $g(1)=0$. Define $f_n(x)=g(x)x^n$. Prove that $f_n \to 0$ uniformly.

Proof of exercise 5.1.9:

$\forall \varepsilon > 0$, we want to find $N(\varepsilon) > 0 \ni \forall x \in [0,1], \forall n \geq N$,

$|f_n(x)-0| = |g(x)x^n| = |g(x)||x^n| \leq \varepsilon.$

Since g is continuous on $[0,1]$, g is continuous at 1.

That is, $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \ni \forall x \in (1-\delta,1]$, $|g(x)| \leq \varepsilon.$

Since $0 \leq 1-\delta \leq x \leq 1$, $|x| \leq 1 \Rightarrow |x|^n \leq 1^n = 1.$

Case I: $x \in (1-\delta,1]$:

$\forall n \in \mathbf{N}$, $|f_n(x)-0| = |g(x)x^n| = |g(x)||x^n| = |g(x)||x|^n \leq |g(x)| \leq \varepsilon.$

Case II: $x \in [1,1-\delta]$

By Theorem 3.2.1, since g is continuous on $[0,1-\delta]$, g is bounded. That is,

$\exists M > 0 \ni |g(x)| \leq M \ \forall x \in [0,1-\delta].$

Since $0 \leq x \leq 1-\delta$, $|x| \leq |1-\delta|$ and thus $|x|^n \leq |1-\delta|^n$.

Thus in order to ensure that:

$|f_n(x)-0| = |g(x)x^n| = |g(x)||x^n| = |g(x)||x|^n \leq M|x|^n \leq M|1-\delta|^n \leq \varepsilon$, we must have:

$M|1-\delta|^n \leq \varepsilon \Leftrightarrow \ln M + n\ln|1-\delta| \leq \ln\varepsilon \Leftrightarrow n\ln|1-\delta| \leq \ln\varepsilon - \ln M$. Since

$|1-\delta| < 1$, $\ln|1-\delta| < 0$. Thus, $n \geq \dfrac{\ln\varepsilon - \ln M}{\ln|1-\delta|}$. Choose $N = \left\lceil \dfrac{\ln\varepsilon - \ln M}{\ln|1-\delta|} \right\rceil$. Then,

$\forall n \geq N$, $|f_n(x)-0| \leq M|1-\delta|^{\frac{\ln\varepsilon - \ln M}{\ln|1-\delta|}} = \varepsilon$. Thus, $f_n \to 0$ uniformly on $[0,1]$.

Q.E.D.