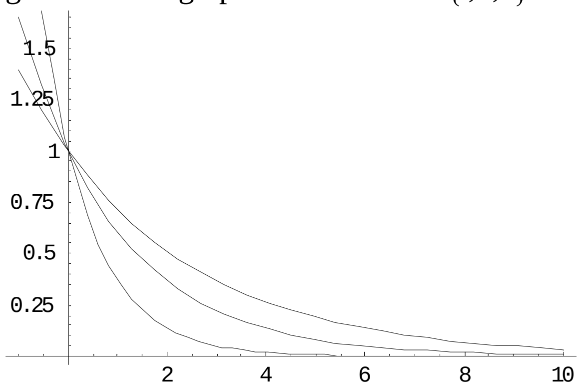
5.1.8 Let $f_n(x) = \frac{x}{n}e^{-\frac{x}{n}}$. Prove that $f_n \to 0$ pointwise but not uniformly on $[0,\infty)$.

Proof of exercise 5.1.8:

$\forall \varepsilon > 0,\ \forall x \in [0,\infty)$, we want to find $N(\varepsilon, x) > 0 \ni \forall n \geq N$,

$\left|\frac{x}{n}e^{-\frac{x}{n}} - 0\right| \leq \varepsilon.$

A quick glance of the graph of $e^{-\frac{x}{n}}$ for $n \in \{1,2,3\}$ is shown:



Clearly, $\forall x \in [0,\infty),\ e^{-\frac{x}{n}} \leq 1.$

Thus, we have:

$\left|\frac{x}{n}e^{-\frac{x}{n}}\right| = \left|\frac{x}{n}\right|\left|e^{-\frac{x}{n}}\right| \leq \left|\frac{x}{n}\right| = \frac{|x|}{n} \leq \varepsilon$ whenever $n \geq \frac{|x|}{\varepsilon}$. Choose $N(\varepsilon, x) = \frac{|x|}{\varepsilon}$. Then,

$\forall n \geq N,\ \left|\frac{x}{n}e^{-\frac{x}{n}}\right| = \left|\frac{x}{n}\right|\left|e^{-\frac{x}{n}}\right| \leq \left|\frac{x}{n}\right| \leq \frac{|x|}{n} \leq \frac{|x|\varepsilon}{|x|} = \varepsilon.$ Thus, $f_n \to 0$ pointwise.

But, does this convergence occur uniformly? Given any $\varepsilon > e^{-1}$ and $n \in \mathbf{N}$, pick $x = n.$ $\forall n,$ we have that $x \in [0,\infty)$, and:

$\left|\frac{x}{n}e^{-\frac{x}{n}}\right| = \left|\frac{n}{n}e^{-\frac{n}{n}}\right| = \left|e^{-1}\right| > \varepsilon.$ Thus, $\exists \varepsilon > 0$ for which $\forall n \in \mathbf{N},\ \exists x \in [0,\infty) \ni$

$\left|f_n(x) - 0\right| > \varepsilon. \Rightarrow$ f_n does not converge uniformly to 0. Since it does converge pointwise to zero, it cannot converge uniformly to any other function. Thus, it does not converge uniformly.