5.1.7 Let $f_n(x) = \frac{nx}{1+n^2x^2}$. Prove that $f_n \to 0$ pointwise but not uniformly on $[0,1]$.

Proof of 5.1.8:

$\forall \varepsilon > 0,\ \forall x \in [0,1],$ we want to find $N(\varepsilon, x) > 0 \ni \forall n \geq N$,

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| > \varepsilon.$$

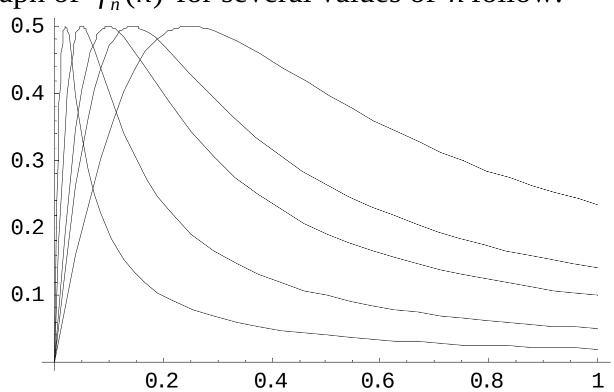
First, recognize that, since $n \geq 1, x \in [0,1]$, we have $\frac{1}{nx} + nx \geq nx$

$$\Rightarrow \frac{1}{\frac{1}{nx} + nx} \leq \frac{1}{nx}.$$

$$\left| \frac{nx}{1+n^2x^2} \right| = \left| \frac{1}{\frac{1}{nx} + nx} \right| \leq \left| \frac{1}{nx} \right| = \frac{1}{n}\left| \frac{1}{x} \right| \leq \varepsilon$$ whenever $n \geq \frac{1}{\varepsilon|x|}$. Thus, pick

$N(\varepsilon, x) = \left\lceil \frac{1}{\varepsilon|x|} \right\rceil$. $\Rightarrow f_n \to 0$ pointwise on $[0,1]$.

The graph of $f_n(x)$ for several values of n follow:



Clearly, these functions do not appear to be uniformly converging. To see this analytically, fix any $0 < \varepsilon < \frac{1}{2}$ and any $n \geq 1$.

Then, $\exists x \in [0,1] \ni \left| \frac{nx}{1+n^2x^2} \right| > \varepsilon$. In particular, pick $x = \frac{1}{n} \in [0,1]$

$$\Rightarrow \left| \frac{nx}{1+n^2x^2} \right| = \left| \frac{n\left(\frac{1}{n}\right)}{1+n^2\left(\frac{1}{n}\right)^2} \right| = \left| \frac{1}{1+1} \right| = \frac{1}{2} > \varepsilon.$$ Thus, f_n does not converge uniformly to 0.