5.1.6 Prove that $\sin\left(x+\frac{1}{n}\right) \to \sin(x)$ uniformly on **R**.

Proof of exercise 5.2.6:

By the Mean Value Theorem, we have: $\sin\left(x+\frac{1}{n}\right) - \sin(x) = \cos(c(x))\left(\frac{1}{n}\right)$ for some $x \le c(x) \le x+\frac{1}{n}$.

We want to find N satisfying $\forall \varepsilon > 0$, $\forall n \ge N$, $\forall x \in \mathbf{R}$,

$$\left|f_n(x) - f(x)\right| = \left|\sin\left(x+\frac{1}{n}\right) - \sin(x)\right| = \left|\cos(c(x))\left(\frac{1}{n}\right)\right| = \frac{\left|\cos(c(x))\right|}{n} \le \frac{1}{n} \le \varepsilon$$

Pick $N \ge \frac{1}{\varepsilon}$. Then, we have:

$$\left|f_N(x) - f(x)\right| = \left|\sin\left(x+\frac{1}{N}\right) - \sin(x)\right| = \frac{\left|\cos(c(x))\right|}{N} \le \frac{1}{N} \le \varepsilon.$$

Therefore, $\sin\left(x+\frac{1}{n}\right) \to \sin(x)$ uniformly on **R.** Q.E.D.