

5.1.5 Prove that $f_n(x) = \left(x - \frac{1}{n}\right)^2$ converges uniformly on any finite interval.

Proof of exercise 5.1.5:

In order to prove uniform convergence, it is necessary first to prove pointwise convergence on the domain. Let $E = [a, b]$. $\forall \varepsilon > 0, \forall x \in [a, b]$, we want to find

$$N(\varepsilon, x) > 0 \ni \forall n \geq N, \left| \left(x - \frac{1}{n}\right)^2 - x^2 \right| \leq \varepsilon.$$

First, recognize that $n \geq 1 \Rightarrow n^2 = nn \geq 1n = n \Rightarrow \frac{1}{n^2} \leq \frac{1}{n}$.

$$\begin{aligned} \left| \left(x - \frac{1}{n}\right)^2 - x^2 \right| &= \left| x^2 - \frac{2}{n}x + \frac{1}{n^2} - x^2 \right| = \left| \frac{1}{n^2} - \frac{2}{n}x \right| \leq \left| \frac{1}{n} - \frac{2}{n}x \right| = \left| \frac{1}{n}(1 - 2x) \right| \\ &= \left| \frac{1}{n} \right| |1 - 2x| = \frac{1}{n} |1 - 2x| \leq \varepsilon. \end{aligned}$$

Whenever $n \geq \frac{|1 - 2x|}{\varepsilon}$, this condition is satisfied. Thus, let $N = \left\lceil \frac{|1 - 2x|}{\varepsilon} \right\rceil$.

Next, it remains to demonstrate uniform convergence:

$\forall \varepsilon > 0$, we want to find $N(\varepsilon) > 0 \ni \forall x \in [a, b], \forall n \geq N$,

$$\left| \left(x - \frac{1}{n}\right)^2 - x^2 \right| \leq \varepsilon.$$

First, recognize that $\forall x \in [a, b], |1 - 2x| \leq \max\{|1 - 2a|, |1 - 2b|\}$.

$$\begin{aligned} \left| \left(x - \frac{1}{n}\right)^2 - x^2 \right| &= \left| x^2 - \frac{2}{n}x + \frac{1}{n^2} - x^2 \right| = \left| \frac{1}{n^2} - \frac{2}{n}x \right| \leq \left| \frac{1}{n} - \frac{2}{n}x \right| = \left| \frac{1}{n}(1 - 2x) \right| \\ &= \left| \frac{1}{n} \right| |1 - 2x| = \frac{1}{n} \max\{|1 - 2a|, |1 - 2b|\} \leq \varepsilon. \end{aligned}$$

Whenever $n \geq \frac{\max\{|1 - 2a|, |1 - 2b|\}}{\varepsilon}$, this condition is satisfied. Thus, let

$N = \left\lceil \frac{\max\{|1 - 2a|, |1 - 2b|\}}{\varepsilon} \right\rceil$. Therefore, $\left(x - \frac{1}{n}\right)^2 \xrightarrow{n \rightarrow \infty} x^2$ uniformly on $[a, b]$.