Michael Powell; 603-210-926
Math 131B; Prof. Eskin
University of California at Los Angeles
January 21st, 2005
Problem Set 2

5.1.1 Prove that the sequence $f_n(x) = x^n$ converges uniformly to zero on any interval of the form $[0, \mu]$ if $\mu < 1$.

Proof of exercise 5.1.1:

$$\forall \varepsilon > 0,\ \exists\ N = N(\varepsilon, x) = \left\lceil \frac{\ln \varepsilon}{\ln|x|} \right\rceil > 0 \ni \forall n \geq N,$$

$\left| f_n(x) - 0 \right| = \left| f_n(x) \right| = \left| x^n \right| = |x|^n \leq |x|^{\frac{\ln \varepsilon}{\ln|x|}} = \varepsilon$ where $\left| x^n \right| = |x|^n$ by proposition 1.1.2.b. Since this holds $\forall x \in [0, \mu]$, it follows that $f_n$ converges pointwise to 0 on $[0, \mu]$.

Since $0 \leq x \leq \mu$, we have that $|x|^n \leq |\mu|^n$. Thus, if we pick $N(\varepsilon) = \left\lceil \frac{\ln \varepsilon}{\ln|\mu|} \right\rceil$, we have:

$\left| f_n(x) - 0 \right| = \left| x^n \right| = |x|^n \leq |\mu|^n$. Taking $n \geq N(\varepsilon)$,

$|\mu|^n \leq |\mu|^{\frac{\ln \varepsilon}{\ln|\mu|}} = \varepsilon$. Thus, $\forall n \geq N(\varepsilon), \left| f_n(x) - 0 \right| \leq \varepsilon$ and $f_n$ converges uniformly to 0.