4.6.7 Show that the partial derivatives of

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0\end{cases}$$

exist at $(0,0)$ but that *f* is not differentiable there.

Answer to exercise 4.6.7:

Fix $x=0$ and define $g(y)=f(0,y)=\dfrac{(0)(y)}{\sqrt{0^2+y^2}}=\dfrac{0}{|y|}=0$

$$\frac{dg(0)}{dy}=\frac{\partial f(0,0)}{\partial y}=0 \text{ since } g(y) \text{ is constant } (=0)\ \forall y\in\mathbf{R}.$$

Fix $y=0$ and define $h(x)=f(x,0)=\dfrac{(x)(0)}{\sqrt{x^2+0^2}}=\dfrac{0}{|x|}=0$

$$\frac{dh(0)}{dx}=\frac{\partial f(0,0)}{\partial x}=0 \text{ since } h(x) \text{ is constant } (=0)\ \forall x\in\mathbf{R}.$$

That is, $f_x(0,0)=0=f_y(0,0)$, i.e. both partial derivatives exist.

But is $f(x,y)$ differentiable? By definition, it suffices to ask if $f_x(x,y)$ is continuous at $(0,0)$.

Suppose we approach $(0,0)$ along the line $y=kx$, i.e. let $x_n=\dfrac{1}{n},\ y_n=\dfrac{k}{n}$.

$$f_x(x,y)=\frac{(x^2+y^2)^{1/2}y}{x^2+y^2}-\frac{xy(1)(2x)}{2(x^2+y^2)^{1/2}(x^2+y^2)^2}=\frac{(x^2+y^2)y-x^2y}{(x^2+y^2)^{3/2}}$$

$$=\frac{x^2y+y^3-x^2-y}{(x^2+y^2)^{3/2}}=\frac{y^3}{(x^2+y^2)^{3/2}}$$

Since $(x_n,y_n)\to(0,0)$ as $n\to\infty$, $f_x(x,y)$ is continuous iff

$f_x(x,y)\to f_x(0,0)=0$ as $n\to\infty$

$$f_x(x_n,y_n)=\frac{\left(\frac{k}{n}\right)^3}{\left(\left(\frac{1}{n}\right)^2+\left(\frac{k}{n}\right)^2\right)^{3/2}}=\frac{\frac{k^3}{n^3}}{\left[\left(\frac{1}{n}\right)^2(1+k^2)\right]^{3/2}}=\frac{\frac{k^3}{n^3}}{\frac{1}{n^3}(1+k^2)^{3/2}}=\frac{k^3}{(1+k^2)^{3/2}},$$

which does not converge to 0.

Thus, $f_x(x,y)$ is not continuous at $(0,0)\Rightarrow f(x,y)$ is not differentiable at $(0,0)$.