4.6.6 Give an example of a noncompact set $E \subseteq \mathbf{R}^2$ and a continuous function *f* on *E* so that none of the three properties $(a), (b),$ or (c) in theorem 4.6.1 is true.

Answer to exercise 4.6.6:

Let $E = (0,1] \times (0,1]$ and $f : (0,1] \times (0,1] \to \mathbf{R}$ with $(x, y) \mapsto \dfrac{1}{x+y}$.

f is clearly continuous on *E*.

(a) Is *f* bounded? No. $\forall M > 0\ \exists\, (x, y) \in E$ satsifying $\dfrac{1}{x+y} > M$. Pick $x + y < \dfrac{1}{M}$, which is always possible, and this is clear.

(b) Does $\exists\, c \in E \ni f(c) = \sup_{p \in R} f(p)$? No. Since *f* is not bounded from above, $\sup_{p \in R} f(p) = \infty$. There is no $p = (x, y) \in E \ni \dfrac{1}{x+y} = \infty$. Thus there is no $c \in E \ni f(c) = \sup_{p \in R} f(p)$.

(c) Is *f* uniformly continuous? No. Pick $\varepsilon = 1$. Let $p_n = \left(\dfrac{1}{4n}, \dfrac{1}{4n}\right)$ and $\overline{p}_n = \left(\dfrac{1}{4n+4}, \dfrac{1}{4n+4}\right)$. Clearly, $p_n, \overline{p}_n \in E\ \forall n$. $\forall \delta > 0$, we can find $N(\delta)$ satisfying $\|p_n - \overline{p}_n\| = \sqrt{\left(\dfrac{1}{4n} - \dfrac{1}{4n+4}\right)^2 + \left(\dfrac{1}{4n} - \dfrac{1}{4n+4}\right)^2}$

$$= \sqrt{2\left(\frac{1}{4n} - \frac{1}{4n+4}\right)^2} = \sqrt{\left(\frac{4}{4n} - \frac{4}{4n+4}\right)^2} = \sqrt{\left(\frac{1}{n} - \frac{1}{n+1}\right)^2}$$

$$= \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{n+1-n}{n(n+1)}\right| = \left|\frac{1}{n(n+1)}\right| \le \frac{1}{n^2} < \delta \quad \forall n \ge N(\delta)$$

But $|f(p_n) - f(\overline{p}_n)| = \left|\dfrac{1}{\frac{1}{2n}} - \dfrac{1}{\frac{1}{2n+2}}\right| = |2n - 2n - 2| = |-2| = 2 > 1 = \varepsilon.$

Thus, $\exists \varepsilon > 0 \ni \forall \delta > 0, \exists\, p, \overline{p} \in E$ satsifying $\|p - \overline{p}\| < \delta$ but $|f(p) - f(\overline{p})| \ge \varepsilon.$

i.e *f* is not uniformly continuous.