4.6.5.a Prove that $E \subset \mathbf{R}^2$ closed and bounded $\Rightarrow E$ compact.
Proof of exercise 4.6.5.a:

Let $E \subset \mathbf{R}^2$ be closed and bounded.

Since E is bounded, by exercise 2.6.10, every sequence $\{p_n\} \subset E$ has a subsequence $\{p_{n_k}\}$ which converges. That is, $\exists\ p \ni \forall \varepsilon > 0\ \exists K_1(\varepsilon) > 0$ $\ni \forall k \geq K_1, \left\| p_{n_k} - p \right\| < \dfrac{\varepsilon}{2}$ and $\exists K_2(\varepsilon) > 0 \ni \forall\ l \geq K_2, \left\| p - p_{n_l} \right\| < \dfrac{\varepsilon}{2}$.

Then, $\forall k, l \geq \max\{K_1, K_2\}$, we have:

$$\left\| p_{n_k} - p_{n_l} \right\| = \left\| p_{n_k} - p + p - p_{n_l} \right\| \leq \left\| p_{n_k} - p \right\| + \left\| p - p_{n_l} \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

That is, $\{p_{n_k}\}$ is a Cauchy sequence. Since E is closed, $p \in E$.

Therefore, every sequence in E contains a subsequence which converges to a point in E. (i.e. E is compact). Q.E.D.

4.6.5.b Show that if E is not closed and bounded, then E is not compact.
Answer to exercise 4.6.5.b:

Suppose E is not closed. $\exists \{p_n\} \subset E$ Cauchy $\ni p_n \to p \notin E$ as $n \to \infty$.

Since $p_n \to p$ as $n \to \infty$, every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ also converges to $p \notin E$. Thus, $\exists$ a sequence in E which has no subsequences converging to any point in E. (i.e. E is not compact)

Suppose E is not bounded. $\forall n \in \mathbf{N}\ \exists\ p_n \in E \ni \left\| p_n \right\| > n$

Consider the sequence $\{p_n\}$. Any subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfies $\left\| p_{n_j} \right\| > n_j\ \forall j$.

Suppose $p_{n_j} \to p \in \mathbf{R}^2$ as $j \to \infty$ with $\left\| p \right\| \leq N_0 < \infty$ (i.e. assume p_{n_j} has a finite limit)

Take $\varepsilon = 1$. Then $\exists J \in \mathbf{N} \ni \forall j \geq J, \left\| p_{n_j} - p \right\| < 1$. However, we can take $j \ni n_j > N_0 + 10$. Then $\left\| p_{n_j} - p \right\| > 10$ which is not < 1. Thus, $\{p_{n_j}\}$ diverges to $+\infty$.

$\{p_{n_j}\}$ does not converge, but since $\{p_{n_j}\}$ was an arbitrary subsequence of $\{p_n\}$, it follows that $\{p_n\}$ has no convergent subsequences. Thus, $\exists \{p_n\} \subset E$ with no convergent subsequences $\Rightarrow E$ is not compact.

4.6.5.c Is $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ compact?
Answer to exercise 4.6.5.c:

$C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ is bounded since $\forall (x, y) \in C, \left\| x - y \right\| \leq 2$

C is also closed. Take a sequence $\{(x_n, y_n)\} \subset C \ni (x_n, y_n) \to (x_0, y_0)$

Since $(x_n, y_n) \to (x_0, y_0)$, $x_n \to x_0$ and $y_n \to y_0$. Since $(x_n, y_n) \in C\ \forall n$,
$x_n{}^2 + y_n{}^2 \le 1 \Rightarrow \lim_{n\to\infty}\left(x_n{}^2 + y_n{}^2\right) = \lim_{n\to\infty} x_n{}^2 + \lim_{n\to\infty} y_n{}^2 = x_0{}^2 + y_0{}^2 \le \lim_{n\to\infty} 1 = 1$
Thus, $(x_0, y_0) \in C$ and C is closed.
$\Rightarrow$ C is compact by exercise 4.6.5.a.

4.6.5.d Is $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$ compact?
Answer to exercise 4.6.5.d:

No. It is not closed. Take $\{(x_n, y_n)\} = \left\{\left(1 - \frac{1}{n}, 0\right)\right\}$. Then, $\forall n$, $(x_n, y_n) \in C$, but the limit $\lim_{n\to\infty}\left(1 - \frac{1}{n}, 0\right) = (1,0)$ is not in C since $1^2 + 0 = 1 \ge 1$.
$\Rightarrow$ C is not compact by exercise 4.6.5.b since it is not closed.

4.6.5.e Is $E = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \ge 1\}$ compact?

No. E is not bounded since $\forall M > 0\ \exists\ (x, y) \in E \ni \|x - y\| \ge M$. Pick $M_n = n$ and $\{(x_n, y_n)\} = \{(n,0)\}$. $\forall n$, $\|n\| \ge n$.
$\Rightarrow$ C is not compact by exercise 4.6.5.b since it is not bounded.