

4.6.3 Let f be a continuous, real-valued function on a closed rectangle R in the plane. Then f is uniformly continuous. Prove this.

Proof of exercise 4.6.3:

Suppose f is not uniformly continuous. Then $\exists \varepsilon_0 > 0 \ni$ there is no $\delta > 0$

satisfying $\|p - c\| < \delta \Rightarrow |f(p) - f(c)| < \varepsilon_0$

Let $\delta_n = 2^{-n}$. $\exists \{p_n\}, \{c_n\} \ni \forall n, \|p_n - c_n\| < \delta_n = 2^{-n}$ and $|f(p_n) - f(c_n)| \geq \varepsilon_0$

Since R is a closed rectangle, by exercise 2.6.11 \exists a subsequence $\{p_{n_k}\}$ which converges to some $d \in R$.

In addition, we have:

$$\begin{aligned} \|c_{n_k} - d\| &= \|c_{n_k} - p_{n_k} + p_{n_k} - d\| \leq \|c_{n_k} - p_{n_k}\| + \|p_{n_k} - d\| \\ &\leq 2^{-n_k} + \|p_{n_k} - d\| \end{aligned}$$

We know that $\|p_{n_k} - d\| \rightarrow 0$ as $k \rightarrow \infty$ and $2^{-n_k} \rightarrow 0$ as $k \rightarrow \infty$. Thus,

$$\|c_{n_k} - d\| \rightarrow 0 \text{ as } k \rightarrow \infty \text{ or } c_{n_k} \rightarrow d \text{ as } k \rightarrow \infty$$

By continuity of f , $\lim_{k \rightarrow \infty} f(p_{n_k}) = f(\lim_{k \rightarrow \infty} p_{n_k}) = f(d)$ and

$$\lim_{k \rightarrow \infty} f(c_{n_k}) = f(\lim_{k \rightarrow \infty} c_{n_k}) = f(d)$$

That is, $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N}, M(\varepsilon) \in \mathbf{N} \ni |f(p_{n_k}) - f(d)| < \frac{\varepsilon}{2}$ and

$$|f(d) - f(c_{n_k})| < \frac{\varepsilon}{2} \quad \forall n_k \geq \max\{N, M\}.$$

$$\begin{aligned} \Rightarrow |f(p_{n_k}) - f(c_{n_k})| &= |f(p_{n_k}) - f(d) + f(d) - f(c_{n_k})| \\ &\leq |f(p_{n_k}) - f(d)| + |f(d) - f(c_{n_k})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Choose $\varepsilon = \varepsilon_0$ and we have a contradiction $\rightarrow \leftarrow$. Therefore, f must be uniformly continuous on R . Q.E.D.