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Problem Set 2

4.6.2 Let f be a continuous, real-valued function on a closed rectangle R in the plane. Then, $\exists\, c, d \in R \ni f(c) = \sup_{p \in R} f(p),\ f(d) = \inf_{p \in R} f(p)$. Prove this.

Proof of exercise 4.6.2:

By theorem 4.6.1.a, we know that f is bounded on R. Thus, both $\sup_{p \in R} f(p)$, and $\inf_{p \in R} f(p)$ exist and are finite.

Let $M = \sup_{p \in R} f(p)$. By definition of sup, $\exists\ p_n = (x_n, y_n) \in R$ satisfying $M - \frac{1}{n} \le f(p_n) \le M\ \forall n$. (If there was no such point, then $M - \frac{1}{n}$ would be the sup.)
By exercise 2.6.11, $p_n = (x_n, y_n)$ has a subsequence $p_{n_k} = (x_{n_k}, y_{n_k})$ which converges to a point c $\in R$.
By continuity of f, it follows that $\lim_{k \to \infty} f(p_{n_k}) = f(\lim_{k \to \infty} p_{n_k}) = f(c)$
But we also have: $M - \frac{1}{n_k} \le f(p_{n_k}) \le M\ \forall k$ and thus by proposition 2.2.2, since $\lim_{k \to \infty} M - \frac{1}{n_k} = M = \lim_{k \to \infty} M$, $\lim_{k \to \infty} f(p_{n_k}) = M$.
By exercise 2.1.5, a sequence can have at most one limit $\Rightarrow f(c) = M = \sup_{p \in R} f(p)$.

Let $N = \inf_{p \in R} f(p)$. By definition if inf, $\exists\ p_n = (x_n, y_n) \in R$ satisfying $N \le f(p_n) \le N + \frac{1}{n}\ \forall n$. (If there was no such point, $N + \frac{1}{n}$ would be the inf)
By exercise 2.6.11, $p_n = (x_n, y_n)$ has a subsequence $p_{n_k} = (x_{n_k}, y_{n_k})$ which converges to a point $d \in R$.
By continuity of f, it follows that $\lim_{k \to \infty} f(p_{n_k}) = f(\lim_{k \to \infty} p_{n_k}) = f(d)$
But we also have: $N \le f(p_{n_k}) \le N + \frac{1}{n_k}\ \forall k$ and thus by proposition 2.2.2, since $\lim_{k \to \infty} N = N = \lim_{k \to \infty} N + \frac{1}{n_k}$, $\lim_{k \to \infty} f(p_{n_k}) = N$
By exercise 2.1.5, a sequence can have at most one limit $\Rightarrow f(d) = N = \inf_{p \in R} f(p)$

Q.E.D.