4.6.11 Let f be continuously differentiable on the plane and define

$$h(x,y)=\int_0^{f(x,y)} G(t)dt$$

where G is a continuous function on **R**. Prove that h is continuously differentiable and compute its partial derivatives.

Proof of exercise 4.6.11:

Define $F(z)$ by $F(z)=\int_0^z G(t)dt$. Since G is continuous, F is continuously differentiable on $[0,z]$ and $F'(z)=G(z)$ by the Fundamental Theorem of Calculus, part II.

$$\int_0^{f(x,y)} G(t)dt = F(f(x,y)) = h(x,y).$$

By the chain rule, we have:

$\dfrac{\partial h(x,y)}{\partial x}=\dfrac{dF}{df(x,y)}\dfrac{\partial f(x,y)}{\partial x}=G(f(x,y))f_x(x,y)$ by the Fundamental Theorem of Calculus, part II, since G is continuous on **R.**

By assumption, $G(t)$ is continuous on **R**. Thus, it is continuous at $f(x,y)$. Also, $f(x,y)$ was assumed to be continuously differentiable on $\mathbf{R}^2$, so f_x is continuous. $G\circ f$ is continuous since it is the composition of continuous functions.

Since $\dfrac{\partial h(x,y)}{\partial x}=G(f(x,y))f_x(x,y)$ is the product of continuous functions, by theorem 3.1.1, $\dfrac{\partial h(x,y)}{\partial x}$ is continuous.

Therefore, $h(x,y)$ is continuously differentiable.

An analogous argument establishes that $h(x,y)$ is continuously differentiable and

$$\frac{\partial h(x,y)}{\partial y}=\frac{dF}{df(x,y)}\frac{\partial f(x,y)}{\partial y}=G(f(x,y))f_y(x,y). \quad \text{Q.E.D.}$$