4.1.11 Suppose that $f(x) \geq 0$ for all $x \in \mathbf{R}$. Assume that $f(x)^2$ is differentiable. Is $f(x)$ necessarily differentiable?

Counterexample to exercise 4.1.11:

This claim is not true. Here is a counterexample.

Consider the function $f(x) = |x|$. Clearly, $f(x) \geq 0$ for all $x \in \mathbf{R}$. By example 3, we know that $f(x)^2 = (|x|)^2 = x^2$ is differentiable for all $x \in \mathbf{R}$. ($(f(x)^2)' = 2x$).

Let $x = 0$. Consider the sequences $h_n^{(1)} = \frac{1}{n}$ and $h_n^{(2)} = -\frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{f(h_n^{(1)}) - f(0)}{h_n^{(1)}} = \lim_{n\to\infty} \frac{1/n}{1/n} = 1$, but $\lim_{n\to\infty} \frac{f(h_n^{(2)}) - f(0)}{h_n^{(2)}} = \lim_{n\to\infty} \frac{-1/n}{1/n} = -1$.

Therefore, the limit does not exist, and $f(x)$ is not differentiable.

Therefore, we have a function f such that f^2 is differentiable at a point but f is not.