3.3.4 Let $f(x)=3x$ on the interval $[0,1]$ and let $\varepsilon > 0$ be given.

3.3.4.a Explain how to construct explicitly a partition P so that $U_P(f)-L_P(f)\le\varepsilon$.

Answer to exercise 3.3.4.a:

Let $\{P_k\}$ be a sequence of partitions of $[0,1]$. For each k, denote the points of the partition P_k by $\frac{0}{k}<\frac{1}{k}<...<\frac{k}{k}$. Since the length of each interval is $\frac{1}{k}$, it follows that as $k\to\infty$, $\Delta(P_k)\to 0$.

$U_{P_k}(3x)=\sum_{i=1}^{k}\frac{3i}{k}\left(\frac{i}{k}-\frac{i-1}{k}\right)=\sum_{i=1}^{k}\frac{3i}{k^2}$; $L_{P_k}(3x)=\sum_{i=1}^{k}\frac{3(i-1)}{k}\left(\frac{i}{k}-\frac{i-1}{k}\right)=\sum_{i=1}^{k}\frac{3(i-1)}{k^2}$

Therefore, $U_{P_k}(3x)-L_{P_k}(3x)=\sum_{i=1}^{k}\frac{3i}{k^2}-\frac{3i-3}{k^2}=\sum_{i=1}^{k}\frac{3}{k^2}=\frac{3}{k}$. Let $k\ge\frac{3}{\varepsilon}$. Then we have that $U_{P_k}(3x)-L_{P_k}(3x)\le\varepsilon$.

3.3.4.b Compute $\int_0^1 f(x)dx$ without using the Fundamental Theorem of Calculus.

Answer to exercise 3.3.4.b:

Construct $\{P_k\}$ as above. By corollary 3.3.2, we can choose any Riemann sum corresponding to P_k and $S_{P_k}(3x)\to\int_0^1 3xdx$ as $k\to\infty$. Choose the Riemann sum $U_{P_k}(3x)$ for this matter.

$U_{P_k}(3x)=\sum_{i=1}^{k}\frac{3i}{k^2}=\frac{3}{k^2}\sum_{i=1}^{k}i=\frac{3}{k^2}\cdot\frac{k(k+1)}{2}=\frac{3k^2+3k}{2k^2}=\frac{3+\frac{3}{k}}{2}\to\frac{3}{2}$ as $k\to\infty$ by theorem 2.2.6. Therefore, we have that $\int_0^1 3xdx=\frac{3}{2}$.