3.3.15 Let f be a continuous function on the interval $[a,b]$. Prove that there exists an $x \in [a,b]$ so that

$$f(x) = \frac{1}{b-a}\int_a^b f(x)dx$$

Why does this make sense geometrically?

Proof of exercise 3.3.15:

Define $\frac{1}{b-a}\int_a^b f(x)dx = M$. In order to show that $\exists\ x^* \in [a,b]$ satisfying $f(x^*) = M$, it suffices to show that $\exists\ c \in [a,b]$ satisfying $f(c) > M$ and $\exists\ d \in [a,b]$ satisfying $f(d) < M$. Then, since f is a continuous function on a closed interval, by the intermediate value theorem, such an x must exist.

In order to get a contradiction, assume that $\forall x \in [a,b]$, $f(x) > M$. By exercise 3.2.4, we have that $\exists \alpha > 0 \ni f(x) > M + \alpha\ \forall x \in [a,b]$. By a slight extension of theorem 3.3.4 and theorem 3.3.3, we then have that $M = \frac{1}{b-a}\int_a^b f(x)dx \geq \frac{1}{b-a}\int_a^b (M+\alpha)dx = \frac{1}{b-a}\int_a^b Mdx + \frac{1}{b-a}\int_a^b \alpha dx = M + \frac{\alpha}{b-a} > M$, which is a contradiction. Therefore, $\exists\ c \in [a,b] \ni f(c) \leq M$. If $f(c) = M$, let $x^* = c$ and we are done. Otherwise, $f(c) < M$ and we must still find some $d \in [a,b] \ni f(d) > M$.

Assume that $\forall x \in [a,b]$, $f(x) < M$. By exercise 3.2.4, we have that $\exists\ \alpha > 0 \ni f(x) < M - \alpha\ \forall x \in [a,b]$. By a slight extension of theorem 3.3.4 and theorem 3.3.3, we then have that $M = \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{b-a}\int_a^b (M-\alpha)dx$ $= \frac{1}{b-a}\int_a^b Mdx - \frac{1}{b-a}\int_a^b \alpha dx = M - \frac{\alpha}{b-a} < M$, which is a contradiction. Therefore, $\exists\ d \in [a,b] \ni f(d) \geq M$. If $f(d) = M$, let $x^* = d$ and we are done. Otherwise, $f(d) > M$

Since $\exists\ c,d \in [a,b]$ satisfying $f(c) < M$ and $f(d) > M$ it follows that $\exists\ x^*$ between c and d satisfying $f(x^*) = M = \frac{1}{b-a}\int_a^b f(x)dx$. Q.E.D.

Geometrically, this is equivalent to saying that a continuous function on an interval $[a,b]$ takes on its average value at some point in that interval, which makes sense since the function cannot be strictly greater than its average value and it cannot be strictly below its average value, because, being continuous, it must obey the intermediate value theorem.