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Problem Set 6

3.3.1 Let f be the function on $[0,1]$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Explain why $U_P(f) = 1$ and $L_P(f) = 0$ for every partition P. Is f Riemann integrable?

Lemma 3.3.1.a:

There is an irrational number between any two rational numbers.

Proof of lemma 3.3.1a:

Take $q, q' \in \mathbf{Q}$. Without loss of generality, assume that $q < q'$.

Then we have that $q' - q > 0 \Rightarrow q' - q > (q' - q)(2 - \sqrt{2}) > 0$ since $0 \le 2 - \sqrt{2} \le 1$.

Therefore, $\exists\ x = q + (q' - q)(2 - \sqrt{2})$ satisfying $q < x < q'$.

Suppose x is rational. Then $\dfrac{2(q'-q) - (2-\sqrt{2})(q'-q)}{q'-q} = 2 - 2 + \sqrt{2} = \sqrt{2}$ is rational since it is the sum/product/quotient of rational numbers. But this is a contradiction, since we know that $\sqrt{2}$ is irrational. $\rightarrow\leftarrow$ Therefore, it must be that x is irrational. Q.E.D.

Lemma 3.3.1.b:

The interval $[a,b]$ contains both an irrational and a rational number.

Proof of lemma 3.3.1.b:

Case I: $a \in \mathbf{Q}, b \in \mathbf{Q}$

By lemma 3.3.1.a, there is some $x \in [a,b]$ irrational.

Case II: $a \notin \mathbf{Q}$, $b \notin \mathbf{Q}$

By exercise 1.1.11, we have that between any two real numbers, there is a rational number.

Case III/IV: $a \notin \mathbf{Q}$ and $b \in \mathbf{Q}$ or $a \in \mathbf{Q}$ and $b \notin \mathbf{Q}$

The lemma is trivially true in this case. Q.E.D.

Answer to exercise 3.3.1:

Consider a general partition $P: 0 = x_0 < x_1 < ... < x_n = 1$ of [0,1]. In any interval $[x_j, x_{j+1}]$, by lemma 3.3.1.b, there is both a rational and an irrational number.

That is, $\exists\ x' \in [x_j, x_{j+1}]$ rational and $\exists\ x'' \in [x_j, x_{j+1}]$ irrational. Thus, $\exists\ x' \in [x_j, x_{j+1}] \ni f(x') = 0$ and $x'' \in [x_j, x_{j+1}] \ni f(x'') = 1$. Therefore,

$M_j = 1\ \forall j$ and $m_j = 0\ \forall j$

$$\Rightarrow L_P(f) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}) = \sum_{j=1}^{n} 0(x_j - x_{j-1}) = 0 \text{ and}$$

$$U_P(f) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = \sum_{j=1}^{n} (x_j - x_{j-1}) = 1$$

Define $P([0,1]) = \{\textit{Partitions of } [0,1]\}$. Since this holds $\forall P \in P([0,1])$, it follows that $\sup_P \{L_P(f)\} = 0$ and $\inf_P \{U_P(f)\} = 1$. In particular, $\sup_P \{L_P(f)\} \neq \inf_P \{U_P(f)\}$. Therefore, f is not Riemann integrable.