3.2.11 Show by example that a function can be uniformly continuous without being Lipschitz continuous. Hint: consider $f(x)=\sqrt{x}$.

Answer to exercise 3.2.11:

Suppose $S=[0,1]$. Let $f(x)=\sqrt{x}$.

By exercise 3.1.11.b, we know that f is continuous at $x=0$. By exercise 3.1.11.a, we know that f is continuous at all $x\in(0,1]$. Therefore, f is continuous on $[0,1]$. Since $[0,1]$ is a closed interval, by theorem 3.2.5, we have that f is uniformly continuous on $[0,1]$.

But is f Lipschitz continuous on $[0,1]$?

Take $x,y\in[0,1],\ x\neq y$. $\dfrac{\left|\sqrt{x}-\sqrt{y}\right|}{|x-y|}=\dfrac{\left|\dfrac{x-y}{\sqrt{x}+\sqrt{y}}\right|}{|x-y|}=\dfrac{1}{\left|\sqrt{x}+\sqrt{y}\right|}=\dfrac{1}{\sqrt{x}+\sqrt{y}}$

Let $M>0$ arbitrary. Let $y=0$. Then $\dfrac{1}{\sqrt{x}}>M\ \forall\ x<\left(\dfrac{1}{M}\right)^2$. Thus, there is no $M>0$ such that $\forall x,y\in[0,1],\ x\neq y,\ \dfrac{\left|\sqrt{x}-\sqrt{y}\right|}{|x-y|}\leq M$. i.e. f is not Lipschitz continuous on $[0,1]$.