3.1.11 Let $f(x)=\sqrt{x}$ with domain $\{x \mid x \geq 0\}$.

3.1.11.a Let $\varepsilon > 0$ be given. For each $c > 0$, show how to choose δ so that $|x-c| \leq \delta \Rightarrow |\sqrt{x}-\sqrt{c}| \leq \varepsilon$. Hint: write

$$\sqrt{x}-\sqrt{c}=\frac{x-c}{\sqrt{x}+\sqrt{c}}.$$

Answer to exercise 3.1.11.a:

Let $\delta \leq c$ and take $|x-c| \leq \delta \Rightarrow c-x=|c|-|x| \leq |x-c| \leq \delta$ by exercise 1.1.10.a and using the fact that $\forall x \in D(f),\ x \geq 0$ and $c>0 \Rightarrow x \geq c-\delta \Rightarrow \sqrt{x} \geq \sqrt{c-\delta}$

$$|\sqrt{x}-\sqrt{c}|=\left|\sqrt{x}-\sqrt{c}\left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right)\right|=\left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| \leq \frac{\delta}{|\sqrt{x}+\sqrt{c}|}=\frac{\delta}{\sqrt{x}+\sqrt{c}}$$

$$\leq \frac{\delta}{\sqrt{c-\delta}+\sqrt{c}} \leq \varepsilon \text{ when } \delta \leq \min\{c, \varepsilon\sqrt{c}\}$$

Verification:

Take $\delta \leq c \Rightarrow -\delta \geq -c \Rightarrow c-\delta \geq c-c=0 \Rightarrow \sqrt{c-\delta} \geq 0$

$$\Rightarrow \sqrt{c-\delta}+\sqrt{c} \geq \sqrt{c} \Rightarrow \frac{\delta}{\sqrt{c-\delta}+\sqrt{c}} \leq \frac{\delta}{\sqrt{c}}.$$

Take $\delta \leq \varepsilon\sqrt{c} \Rightarrow \dfrac{\delta}{\sqrt{c}} \leq \dfrac{\varepsilon\sqrt{c}}{\sqrt{c}}=\varepsilon.$

3.1.11.b Give a separate argument to show that f is continuous at zero.

Answer to exercise 3.1.11.b:

Suppose that f is not continuous at zero. That is, we can find some sequence $\{x_n\} \subset D(f) \ni x_n \to 0$ and $f(x_n)$ does not converge to $f(0)=0$.

(i.e. $\forall \varepsilon > 0\ \forall N \in \mathbf{N},\ \exists\, n \geq N \ni |f(x_n)-f(x)| \geq \varepsilon$.)

Take $\varepsilon > 1$. Then $\forall N \in \mathbf{N},\ \exists\, n \geq N \ni |f(x_n)-f(0)|=\sqrt{x_n}>1 \Rightarrow x_n>1$.

Since $x_n \to 0$, $\forall \delta > 0\ \exists N(\delta) \in \mathbf{N} \ni \forall n \geq N,\ |x_n|=x_n \leq \delta$.

Take $\delta \leq 1 \Rightarrow \forall n \geq N(1),\ x_n \leq 1$, which is a contradiction. $\rightarrow\leftarrow$ Therefore, f is continuous at $x=0$.