3.1.ex.1.a Use the information from problem 5 to construction a function $f:\mathbf{R}\to\mathbf{R}$ which is not continuous at any point.

Lemma 3.1.ex.1.a.a:

There is an irrational number between any two rational numbers.

Proof of lemma 3.1.ex.1.a.a:

Take $q, q' \in \mathbf{Q}$. Without loss of generality, assume that $q < q'$.

Then we have that $q'-q > 0 \Rightarrow q'-q > (q'-q)(2-\sqrt{2}) > 0$ since $0 \le 2-\sqrt{2} \le 1$.

Therefore, $\exists\ x = q + (q'-q)(2-\sqrt{2})$ satisfying $q < x < q'$.

Suppose x is rational. Then $\dfrac{2(q'-q)-(2-\sqrt{2})(q'-q)}{q'-q} = 2-2+\sqrt{2} = \sqrt{2}$ is rational since it is the sum/product/quotient of rational numbers. But this is a contradiction, since we know that $\sqrt{2}$ is irrational. $\rightarrow\leftarrow$ Therefore, it must be that *x* is irrational. Q.E.D.

Answer to exercise 3.1.ex.1.a:

Consider the function $f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$, the indicator function for the rationals.

Using the $\delta - \varepsilon$ arguments allowed by theorem 3.1.3, choose $\varepsilon < 1$.

Since **Q** is dense in **R**, we have that $\forall \delta > 0\ \forall x \notin \mathbf{Q},\ \exists\ q \in \mathbf{Q} \ni |x-q| < \delta$.

$\Rightarrow |f(x) - f(q)| = 1 > \varepsilon$. Therefore, we cannot find a δ for $\varepsilon < 1$ and *f* is not continuous at any irrational point.

By lemma 3.1.ex.1.a.a, we have that $\forall \delta > 0\ \forall q \in \mathbf{Q}\ \exists x \notin \mathbf{Q} \ni |x-q| < \delta$

$\Rightarrow |f(x) - f(q)| = 1 > \varepsilon$. Therefore, we cannot find a δ for $\varepsilon < 1$ and *f* is not continuous at any rational point.

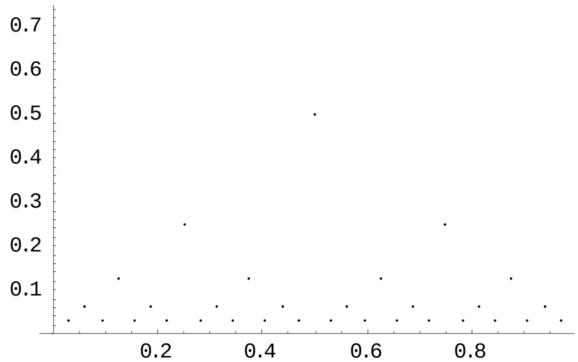
Therefore, $f(x)$ is not continuous at any point. Q.E.D.

3.1.ex.1.b Use a similar (but more clever) method to construct $g:\mathbf{R}\to\mathbf{R}$ which is continuous at all the irrational points and discontinuous (i.e. not continuous) at every rational point.

Answer to exercise 3.1.ex.1.b:

Here is another classic example of a function with some serious discontinuity issues.

Consider the function $f(x) = 0$ if $x \notin \mathbf{Q}$ and $f(x) = \dfrac{1}{b}$ if $x \in \mathbf{Q}$ and $x = \dfrac{a}{b}$ where a and b are relatively prime. This is what it looks like on $[0,1]$:



Consider any rational point $q = \frac{a}{b}$ which is irreducible. $f(q) = \frac{1}{b} > 0$. We also know that the irrational numbers are dense in **R.** (Which I will not go into) Therefore, we can construct a sequence of irrational numbers $\{x_n\} \ni x_n \to q$. Suppose that f is continuous at q. Since $f(x_n) = 0 \ \forall n$, it follows that $f(q) = 0$ by lemma 3.1.5.a:, which is a contradiction. $\rightarrow\leftarrow$ Therefore, f is discontinuous at all rational points.

Let x be an irrational number. Take an arbitrary sequence $\{x_n\}$ which converges to x. We want to show that $f(x_n) \to f(x) = 0$ as $n \to \infty$. Without loss of generality, suppose $x_n \in \mathbf{Q} \ \forall n$. (If for any $\bar{n}$, $x_{\bar{n}} \in \mathbf{R} \setminus \mathbf{Q}$, then $f(x_{\bar{n}}) = 0$, which cannot hurt the convergence of this sequence.)

Let $x_n = \frac{a_n}{b_n} \Rightarrow f(x_n) = \frac{1}{b_n}$. The problem that remains is to show that $\frac{1}{b_n} \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ arbitrary. We want to find some $N \in \mathbf{N} \ni \forall n \geq N, \left|\frac{1}{b_n}\right| < \varepsilon$ or equivalently, we want to find some $N \in \mathbf{N} \ni n \geq N \Rightarrow b_n < \frac{1}{\varepsilon}$.

Consider the set $A = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbf{Z} \ b \leq \frac{1}{\varepsilon} \right\}$. Denote by d the distance between the number x and the set A. That is, $d = \min_{\frac{a}{b} \in A} \left| \frac{a}{b} - x \right|$. Clearly, $d > 0$.

Since $\frac{a_n}{b_n} \to x$, we have that $\exists \ N'(d) \ni \forall n \geq N', \left| \frac{a_n}{b_n} - x \right| < d$. Then we have that $\forall n \geq N', \left| f(x_n) - f(x) \right| = \left| \frac{1}{b_n} \right| < \varepsilon$. i.e f is continuous at each $x \in \mathbf{R} \setminus \mathbf{Q}$.

Q.E.D.