3.1.6 Suppose that the function f is defined only on the integers. Explain why it is continuous.

Answer to exercise 3.1.6:

Given any two integers, $z, z' \in \mathbf{Z}$, $|z - z'| \geq 1$ if $z \neq z'$ and $|z - z'| = 0$ if $z = z'$.

Choose a sequence of integers $\{z_n\}$ which converges to $z \in \mathbf{Z}$. That is,

$\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N,\ |z_n - z| < \varepsilon$. Take $\varepsilon < 1$. Then $\forall n \geq N(1)$, we have that $|z_n - z| < 1 \Rightarrow z_n = z\ \forall n \geq N(1)$.

Then we have that $\forall n \geq N(1), |f(z_n) - f(z)| = |f(z) - f(z)| = 0$. That is,

$f(z_n) \to f(z)$ as $n \to \infty$. i.e. f is a continuous function. Q.E.D.