2.6.1 Prove that if a sequence converges it has exactly one limit point. Is the converse true? Prove it or give a counterexample.

Lemma 2.6.1.a:

Let $a < b$. Then $\exists \delta > 0 \ni a + \delta < b - \delta$.

Proof of lemma 2.6.1.a:

Since $a < b$, we have: $b - a > 0 \Rightarrow b - a > \frac{b-a}{2} > 0$. Take $\delta = \frac{b-a}{4}$.

$\Rightarrow b - a > 2\delta \Rightarrow b - \delta > a + \delta$. Q.E.D.

Proof of exercise 2.6.1: (existence)

Take $\varepsilon > 0$ arbitrary, $N \in \mathbf{N}$ arbitrary. Since $a_n \to a$ as $n \to \infty$, we have that $\forall \varepsilon > 0\ \exists N^*(\varepsilon) \in \mathbf{N} \ni \forall n \geq N^*, |a_n - a| < \varepsilon$.

Take $n \geq \max\{N, N^*\}$. Then $|a_n - a| < \varepsilon$. i.e. a is a limit point of the sequence $\{a_n\}$. Q.E.D.

Proof of exercise 2.6.1: (uniqueness)

Suppose $\exists\ d, a$ which are limit points of the sequence $\{a_n\}$. That is,

$\forall \varepsilon > 0, \forall N \in \mathbf{N}, \exists\ n_1 \geq N \ni |a_{n_1} - a| < \varepsilon$ and $\exists\ n_2 \geq N \ni |a_{n_2} - d| < \varepsilon$.

Without loss of generality, assume that $d < a$.

$\Rightarrow \exists \delta > 0 \ni d + \delta < a - \delta$. (By lemma 2.6.1.a)

Since $a_n \to a$ as $n \to \infty$, we have that $\forall \varepsilon > 0\ \exists N^*(\varepsilon) \in \mathbf{N} \ni \forall n \geq N^*$, $|a_n - a| < \varepsilon$.

Let $\varepsilon = \delta \Rightarrow \forall n \geq N^*(\delta), |a_n - a| < \delta \Rightarrow -\delta < a_n - a < \delta$

$\Rightarrow a - \delta < a_n < a + \delta$

$\Rightarrow d + \delta < a - \delta < a_n$

That is, if we let $\varepsilon = \delta$ and $N = N^*(\delta)$ $\forall n \geq N^*(\delta)$, $a_n \notin (d - \delta, d + \delta)$.

Therefore, d is not a limit point of $\{a_n\}$, which is a contradiction $\rightarrow\leftarrow$.

Therefore, there can be at most one limit point of a convergent sequence. Q.E.D.

Counterexample:

Consider the sequence $a_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases}$

$\{a_n\}$ has only one limit point: $d = 0$, but it does not converge.