2.5.9 Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences and define the sets $A, B,$ and C by $A = \{a_n\}$, $B = \{b_n\}$, and $C = \{a_n + b_n\}$. Prove that $\sup C \le \sup A + \sup B$. Give an example to show that strict inequality may hold.

Proof of exercise 2.5.9 (first part):

Since $\{a_n\}$ is a bounded sequence, it follows that A is a bounded set, which implies that A is bounded above. Thus, $\exists!$ least upper bound of A by theorem 2.5.1. Denote $\sup A = a'$.

Since $\{b_n\}$ is a bounded sequence, it follows that B is a bounded set, which implies that B is bounded above. Thus, $\exists!$ least upper bound of B by theorem 2.5.1. Denote $\sup B = b'$.

Since $\sup A = a'$, $\forall n,\ a_n \le a'$.

Since $\sup B = b'$, $\forall n,\ b_n \le b'$.

$\Rightarrow \forall n,\ a_n + b_n \le a' + b_n \le a' + b'$. Since this holds $\forall n,\ a' + b'$ is an upper bound for C.

By definition of leas upper bound, we then have:

$\sup C = \sup_n \{a_n + b_n\} \le a' + b' = \sup A + \sup B$. Q.E.D.

Answer to exercise 2.5.9 (second part):

Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $c_n = a_n + b_n = (-1)^n + (-1)^{n+1}$
$= (-1)^n(1 + (-1)) = (-1)^n(0) = 0\ \forall n$. Thus, we have:
$\sup(A) = 1$, $\sup(B) = 1$, but $\sup(C) = 0$. i.e. $\sup C = 0 < 2 = \sup A + \sup B$.