2.5.5 Suppose that a is an upper bound for the set S and that there is a sequence of elements $x_n \in S$ such that $x_n \to a$. Prove that a is the least upper bound.

Proof of exercise 2.5.5:

Let a be an upper bound for S. Suppose $\exists \{x_n\} \subset S \ni x_n \to a$. That is,
$\forall \varepsilon > 0 \; \exists \; N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, |x_n - a| < \varepsilon.$
$\Rightarrow -\varepsilon < x_n - a < \varepsilon \Rightarrow a - \varepsilon < x_n < a + \varepsilon$. Since a is an upper bound of S, $x_n \leq a$ $\forall n$. Therefore, $\forall n \geq N(\varepsilon)$, $a - \varepsilon < x_n \leq a$. i.e. $x_n \in (a - \varepsilon, a] \subset [a - \varepsilon, a]$.
Therefore, $\forall \varepsilon > 0$, $\exists \; x \in S \ni x \in [a - \varepsilon, a]$. By exercise 2.5.3, it follows that a is the least upper bound for S. Q.E.D.