

2.5.3 Suppose that a set S of real numbers is bounded and let μ be an upper bound for S . Show that μ is the least upper bound of S if and only if for every $\varepsilon > 0$ there is an elements of S in the interval $[\mu - \varepsilon, \mu]$.

Lemma 2.5.3.a:

If $x \in (a, b)$, $x < b$.

Proof of lemma 2.5.3.a: (almost purely by definition)

$x \in (a, b) \Leftrightarrow x \in \{t \in \mathbf{R} \mid a < t < b\} \Rightarrow x < b$. Q.E.D.

Lemma 2.5.3.b:

If for some $\delta > 0$, $x + \delta < y$, $\exists \delta' > 0 \ni x < y - \delta' < x + \delta$.

Proof of lemma 2.5.3.b:

Suppose $x + \delta < y$. Since $\delta > 0$, we know that $x < x + \frac{\delta}{2} < x + \delta$. Let

$\delta' = y - \left(x + \frac{\delta}{2}\right) > 0$. Then $y - \delta' = x + \frac{\delta}{2}$. Therefore, $x < y - \delta' < x + \delta$. Q.E.D.

Proof of exercise 2.5.3:

(\Rightarrow) Let μ be the least upper bound of S . In order to get a contradiction, suppose $\exists \varepsilon > 0 \ni \forall x \in S, x \notin [\mu - \varepsilon, \mu]$.

We know that $\mathbf{R} = (-\infty, \mu - \varepsilon) \cup [\mu - \varepsilon, \mu] \cup (\mu, \infty)$. Since $\forall x \in S,$

$x \notin [\mu - \varepsilon, \mu]$, it must be that $\forall x \in S, x \in (-\infty, \mu - \varepsilon) \cup (\mu, \infty)$. But we know that μ is an upper bound for S . Therefore, $\forall x \in S, x \leq \mu$.

$\Rightarrow \forall x \in S, x \notin (\mu, \infty)$. Thus we have $\forall x \in S, x \in (-\infty, \mu - \varepsilon)$.

That is, $\forall x \in S, x < \mu - \varepsilon$, by lemma 2.5.3.a $\Rightarrow \mu - \varepsilon$ is an upper bound of S .

Since $\varepsilon > 0, \mu - \varepsilon < \mu$, which contradicts μ being the least upper bound.

(\Leftarrow) Suppose $\forall \varepsilon > 0 \exists a \in S \ni a \in [\mu - \varepsilon, \mu]$. Let μ be an upper bound for S .

Suppose that $\exists \mu' < \mu$ which is also an upper bound of S . Since $\mu' < \mu,$

$\exists \delta \in \mathbf{R} \ni \mu' + \delta < \mu$ by lemma 2.5.2.a. Then by lemma 2.5.3.b

$\exists \delta' > 0 \ni \mu' < \mu - \delta' < \mu + \delta$. Let $\varepsilon = \delta'$. By hypothesis, $\exists a \in S \ni a \in [\mu - \delta', \mu]$.

That is, $a \geq \mu - \delta' > \mu'$, which is a contradiction to μ' being an upper bound.

$\rightarrow \leftarrow$ Therefore, μ must be the least upper bound. Q.E.D.