

2.4.9 Suppose that  $a_1 > 0$  and  $a_{n+1} = a_n + \frac{1}{a_n}$ . Prove that  $\{a_n\}$  diverges to  $+\infty$ . Hint: if

$\{a_n\}$  converges to a nonzero limit, find an equation that would be satisfied by the limit.

Lemma 2.4.9.a:

Suppose that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . Then  $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N,$

$$|a_{n+1} - a_n| < \varepsilon.$$

Proof of lemma 2.4.9.a:

Suppose that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . Then  $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N,$

$$|a_n - a| < \frac{\varepsilon}{2}.$$

$$\Rightarrow |a_{n+1} - a_n| = |a_{n+1} - a + a - a_n| \leq |a_{n+1} - a| + |a - a_n| = |a_{n+1} - a| + |a_n - a|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \text{ Q.E.D.}$$

Lemma 2.4.9.b:

If  $a_n \rightarrow a$  as  $n \rightarrow \infty$ ,  $|a_n| \rightarrow |a|$  as  $n \rightarrow \infty$ .

Proof of lemma 2.4.9.b:

Suppose that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . Then  $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N,$

$$|a_n - a| < \varepsilon. \text{ Then, } \forall n \geq N, \text{ by exercise 1.1.10,}$$

$$||a_n| - |a|| \leq |a_n - a| < \varepsilon. \text{ i.e. } |a_n| \rightarrow |a| \text{ as } n \rightarrow \infty. \text{ Q.E.D.}$$

Proof of exercise 2.4.9:

Suppose for a contradiction that  $\{a_n\}$  converges to a nonzero limit  $a$  (and is thus bounded). Then, by lemma 2.4.9.a, we have that  $\exists N(\varepsilon) > 0 \ni \forall n \geq N,$

$$|a_{n+1} - a_n| = \left| a_n + \frac{1}{a_n} - a_n \right| = \left| \frac{1}{a_n} \right| < \varepsilon \Rightarrow |a_n| > \frac{1}{\varepsilon}.$$

Since this holds  $\forall \varepsilon > 0$ , it holds for  $\varepsilon = \frac{1}{M}$  for any given  $M > 0$ . That is,  $\forall M > 0, \exists N(\varepsilon) > 0 \ni \forall n \geq N,$

$|a_n| > M$ . Therefore, by definition,  $|a_n|$  diverges to  $+\infty$ . In particular, it does not converge.

Since  $\{a_n\}$  converges to  $a$ ,  $|a_n|$  should converge to  $|a|$  by lemma 2.4.9.b. But

$|a_n|$  does not converge. Thus,  $a_n$  does not converge to  $a$ , which is a

contradiction.  $\rightarrow\leftarrow$  Thus,  $\{a_n\}$  does not converge to a nonzero limit.

Since  $a_1 > 0, \frac{1}{a_1} > 0 \Rightarrow a_2 = a_1 + \frac{1}{a_1} > a_1$ . Assume  $a_n > 0$ . Then  $\frac{1}{a_n} > 0$

$\Rightarrow a_{n+1} = a_n + \frac{1}{a_n} > a_n$ . (i.e.  $\{a_n\}$  is a monotone increasing sequence)

By the contrapositive of theorem 2.4.3, since  $\{a_n\}$  does not converge  $\{a_n\}$ , either  $\{a_n\}$  is not monotone or  $\{a_n\}$  is not bounded. Since we know that  $\{a_n\}$  is monotone increasing, it must be the case that  $\{a_n\}$  is not bounded.

Thus, since  $\{a_n\}$  is monotone increasing and not bounded, it must diverge to  $+\infty$ . Suppose it does not. Then we can find some  $M > 0 \ni a_n \leq M \forall n$ . That is,  $\{a_n\}$  is bounded, which is a contradiction. Q.E.D.