2.4.7 Suppose that the terms $\{a_n\}$ satisfy $|a_{n+1} - a_n| \le 2^{-n}$ for all n. Prove that $\{a_n\}$ is a Cauchy sequence.

Lemma 2.4.7.a:

Let $a_n = 2^{-n}$. Then, $a_n \to 0$ as $n \to \infty$.

Proof of lemma 2.4.7.a:

$\forall \varepsilon > 0$, we want to find $N(\varepsilon) \in \mathbf{N} \ni \forall\, n \ge N$, $|2^{-n}| < \varepsilon$

$|2^{-n}| = 2^{-n} = \frac{1}{2^n}$.

Take $\varepsilon > 0$ arbitrary. $\frac{1}{2^n} < \varepsilon$ whenever $n \ge N \Leftrightarrow \frac{1}{\varepsilon} < 2^n$ whenever $n \ge N$

$\Leftrightarrow \ln\left(\frac{1}{\varepsilon}\right) < n\ln(2)$ whenever $n \ge N \Leftrightarrow n > \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}$ whenever $n \ge N$.

Thus, we can pick $N \ge \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}$. Then, $\forall n \ge N$, we have:

$|2^{-n}| = \frac{1}{2^n} < \varepsilon$. i.e $a_n \to 0$ as $n \to \infty$. Q.E.D.

Lemma 2.4.7.b:

Suppose $a < 1$. Then $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$.

Proof of lemma 2.4.7.b:

Define $s_n = \sum_{k=1}^{n} a^k$ as the partial sum of the sequence $\{a^k\}$.

$$s_n = \sum_{k=0}^{n} a^k = \frac{1-a}{1-a}\sum_{k=0}^{n} a^k = \frac{\sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k}{1-a} = \frac{1-a^{n+1}}{1-a}.$$

It can be shown that (by an extension of lemma 2.4.7.a) $a_n = 1 - a^{n+1}$ converges to 1 as $n \to \infty$. Thus $s_n = \frac{a_n}{1-a}$ converges to $\frac{1}{1-a}$ by Theorem 2.2.6.

Therefore, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{k=0}^{n} a^k = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$. Q.E.D.

Proof of exercise 2.4.7:

Take $m > n$. We know that $\forall n$, $|a_{n+1} - a_n| \le 2^{-n}$.

Then, $|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \ldots + a_{n+1} - a_n|$

$\le |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \ldots + |a_{n+1} - a_n| \le 2^{-(m-1)} + 2^{-(m-2)} + \ldots + 2^{-n}$

$$= 2^{-n}\left(1 + ... + 2^{-(m-n-1)}\right) \leq 2^{-n}(1 + \frac{1}{2} + \frac{1}{2^2} + ...) = 2^{-n}\left(\frac{1}{1-\frac{1}{2}}\right) = 2^{-n}(2) = 2^{-n+1}$$

where lemma 2.4.7.b was used in the third-to-last step. From lemma 2.4.7.a, we know that 2^{-n+1} converges. Thus, $\forall \varepsilon > 0, \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall\ n \geq N,\ 2^{-n+1} < \varepsilon$ i.e. $\forall \varepsilon > 0, \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall\ m, n \geq N,\ |a_m - a_n| < 2^{-n+1} < \varepsilon$. i.e. $\{a_n\}$ is a Cauchy sequence. Q.E.D.