2.4.14 Let $\{a_n\}$ be a sequence and suppose that $a_n \to a$. Define the sequence $\{b_n\}$ as below and prove that $b_n \to a$.

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

Lemma 2.4.14.a:

$$\frac{c}{n} \to 0 \text{ as } n \to \infty.$$

Proof of lemma 2.4.14.a:

Take $N > \frac{\varepsilon}{|c|}$. Then $\forall n \geq N$, we have:

$$\left|\frac{c}{n}\right| = \frac{|c|}{n} < \frac{\varepsilon|c|}{|c|} = \varepsilon. \text{ i.e. } \frac{c}{n} \to 0 \text{ as } n \to \infty.$$

Proof of exercise 2.4.14

In this proof, there are a couple separate convergent sequence involved. I will start with some preliminaries regarding these sequences.

We know that $a_n \to a$ as $n \to \infty$. That is, $\forall \varepsilon > 0 \; \exists \; N_1(\varepsilon) \in \mathbf{N} \ni \forall \; n \geq N_1$, we have: $|a_n - a| < \frac{\varepsilon}{2}$.

Let $c = |a_1 + ... + a_{N_1 - 1} - (N_1 - 1)a|$. Then $\forall \varepsilon > 0, \exists \; N_2(\varepsilon) \in \mathbf{N} \ni \forall \; n \geq N_2$, we have: $\left|\frac{c}{n}\right| < \frac{\varepsilon}{2}$ by lemma 2.4.14.a.

In order to prove that $b_n \to a$ as $n \to \infty$, we want to show that $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbf{N}$ $\ni \forall \; n \geq N, |b_n - a| < \varepsilon.$

Take $\varepsilon > 0$ arbitrary. Let $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, we have:

$$|b_n - a| = \left|\frac{a_1 + ... + a_{N_1 - 1}}{n} + \frac{a_{N_1} + ... + a_n}{n} - a\right|$$

$$= \left|\frac{a_1 + ... + a_{N_1 - 1}}{n} + \frac{a_{N_1} + ... + a_n}{n} - \frac{(N_1 - 1 + n - N_1)a}{n}\right|$$

$$= \left|\frac{a_1 + ... + a_{N_1 - 1} - (N_1 - 1)a}{n} + \frac{(a_{N_1} - a) + (a_{N_1 + 1} - a) + ... + (a_n - a)}{n}\right|$$

$$\leq \frac{|a_1 + ... + a_{N_1 - 1} - (N_1 - 1)a|}{n} + \frac{|a_{N_1} - a|}{n} + \frac{|a_{N_1 + 1} - a|}{n} + ... + \frac{|a_n - a|}{n}$$ (triangle inequality)

$$= \frac{|c|}{n} + \frac{1}{n}\left(|a_{N_1} - a| + |a_{N_1 + 1} - a| + ... + |a_n - a|\right) \leq \frac{\varepsilon}{2} + \frac{1}{n}\left(\frac{\varepsilon}{2} + ... + \frac{\varepsilon}{2}\right)$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{(n - N_1)}{n} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{\frac{n}{n} - \frac{N_1}{n}}{\frac{n}{n}}\right) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{1 - \frac{N_1}{n}}{1}\right).$$

Taking limits, (and recognizing that $\left(1 - \frac{N_1}{n}\right) \to 1$ as $n \to \infty$), by theorem 2.2.6, we have:

$|b_n - a| \le \varepsilon$. That is, $b_n \to a$ as $n \to \infty$. Q.E.D.