2.4.10 Let $a_1 = \sqrt{2}$, and let a_n for $n \ge 2$ be defined recursively by the formula

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}.$$

2.4.10.a Prove by induction that $\sqrt{2} \le a_n \le 2 \ \ \forall n$.

Proof of exercise 2.4.10.a:

$(n=1)$: $a_1 = \sqrt{2} \Rightarrow \sqrt{2} \le a_1 \le 2.$ i.e. $(n=1)$ is true.

$(n=2)$: $a_2 = \sqrt{2+\sqrt{a_1}} = \sqrt{2+\sqrt{2}} \Rightarrow {a_2}^2 = 2+\sqrt{2}$

We know that $2 \le 2+\sqrt{2} \le 4$ since $0 \le \sqrt{2} \le 2.$

Thus, $2 \le {a_2}^2 \le 4 \Rightarrow \sqrt{2} \le a_2 \le 2.$ i.e $(n=2)$ is true.

Assume $(n=n)$: $\sqrt{2} \le a_n \le 2 \Rightarrow \sqrt{\sqrt{2}} \le \sqrt{a_n} \le \sqrt{2}$

$(n=n+1)$: $a_{n+1} = \sqrt{2+\sqrt{a_n}} \Rightarrow {a_{n+1}}^2 = 2+\sqrt{a_n} \Rightarrow {a_{n+1}}^2 - 2 = \sqrt{a_n}$

By induction assumption, we have:

$\sqrt{\sqrt{2}} \le \sqrt{a_n} = {a_{n+1}}^2 - 2 \le \sqrt{2} \Rightarrow 2+\sqrt{\sqrt{2}} \le {a_{n+1}}^2 \le 2+\sqrt{2}$

Since $2 \le 2+\sqrt{\sqrt{2}}$ and $2+\sqrt{2} \le 4$, we have:

$2 \le {a_{n+1}}^2 \le 4 \Rightarrow \sqrt{2} \le a_{n+1} \le 2$. i.e $(n=n+1)$ is true. Q.E.D.

2.4.10.b Prove that $\{a_n\}$ is a Cauchy sequence and conclude that $\{a_n\}$ converges.

Proof of exercise 2.4.10.b

$$|a_{n+1} - a_n| = \left|\sqrt{2+\sqrt{a_n}} - \sqrt{2+\sqrt{a_{n-1}}}\right|$$

$$= \left|\frac{\sqrt{2+\sqrt{a_n}}+\sqrt{2+\sqrt{a_{n-1}}}}{\sqrt{2+\sqrt{a_n}}+\sqrt{2+\sqrt{a_{n-1}}}}\left(\sqrt{2+\sqrt{a_n}}-\sqrt{2+\sqrt{a_{n-1}}}\right)\right|$$

$$= \left|\frac{2+\sqrt{a_n}-2-\sqrt{a_{n-1}}}{\sqrt{2+\sqrt{a_n}}+\sqrt{2+\sqrt{a_{n-1}}}}\right| \le \left|\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{2\sqrt{2}}\right| \text{ (since } \sqrt{a_n} > 0 \ \forall n.)$$

$$\le \left|\left(\frac{\sqrt{a_n}+\sqrt{a_{n-1}}}{\sqrt{a_n}+\sqrt{a_{n-1}}}\right)\left(\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{2\sqrt{2}}\right)\right| = \left|\frac{a_n - a_{n-1}}{\left(\sqrt{a_n}+\sqrt{a_{n-1}}\right)2\sqrt{2}}\right| \le \left|\frac{a_n - a_{n-1}}{4\sqrt{2}}\right| \text{ (since}$$

$\sqrt{a_n} + \sqrt{a_{n-1}} \ge 2\sqrt[4]{2} \ge 2$ by part a.)

$$\le \frac{|a_n - a_{n-1}|}{2}.$$

Thus, $|a_{n+1} - a_n| \le \frac{1}{2}|a_n - a_{n-1}| \le \frac{1}{2^2}|a_{n-1} - a_{n-2}| \le \ldots \le \frac{1}{2^{n-1}}|a_2 - a_1|$.

Therefore, we have:

$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \ldots + a_{n+1} - a_n|$

$$\leq |a_m - a_{m-1}| + ... + |a_{n+1} - a_n| \leq \frac{|a_2 - a_1|}{2^{m-2}} + ... + \frac{|a_2 - a_1|}{2^{n-1}} = |a_2 - a_1| \left(\frac{1}{2^{n-1}} + ... + \frac{1}{2^{m-2}} \right)$$

$= \frac{|a_2 - a_1|}{2^{n-1}} \left(1 + \frac{1}{2} + ... \frac{1}{2^{m-n-1}} \right) \leq \frac{|a_2 - a_1|}{2^{n-1}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + ... \right) = \frac{|a_2 - a_1|}{2^n}$ where I used lemma 2.4.7.b (see problem 2.4.7) in the last step.

By lemma 2.4.7.a, $\frac{1}{2^n}$ converges to 0 as $n \to \infty$. Thus, $\forall \varepsilon > 0, \exists\, N(\varepsilon) \in \mathbf{N}$ such that $\forall n \geq N$, $\left| \frac{1}{2^n} \right| < \frac{\varepsilon}{|a_2 - a_1|}$.

$\Rightarrow \forall \varepsilon > 0,\ \exists\ N(\varepsilon) \in \mathbf{N} \ni \forall m > n \geq N,\ |a_m - a_n| \leq \frac{|a_2 - a_1|}{2^n} < \frac{\varepsilon |a_2 - a_1|}{|a_2 - a_1|} = \varepsilon.$

i.e. $\{a_n\}$ is a Cauchy sequence.

Since $\{a_n\}$ is a Cauchy sequence of real numbers, by the axiom of completeness, $\{a_n\}$ converges. Q.E.D.