2.2.8 Let $a_n = \sqrt{n+1} - \sqrt{n}$. Prove that $a_n \to 0$.

Proof of exercise 2.2.8:

$\forall \varepsilon > 0,$ we want to show that $\exists\, N(\varepsilon) > 0 \ni \forall n \geq N,\ \left|\sqrt{n+1} - \sqrt{n}\right| < \varepsilon.$

$\left|\sqrt{n+1} - \sqrt{n}\right| = \left|\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\left(\sqrt{n+1} - \sqrt{n}\right)\right| = \left|\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}\right| = \left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right|.$ Since $n > 0, \sqrt{n+1} > 0 \Rightarrow \left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right| < \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}.$

Take $\varepsilon > 0$ arbitrary. $\frac{1}{\sqrt{n}} < \varepsilon$ whenever $n \geq N \Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n}$ whenever $n \geq N \Leftrightarrow \frac{1}{\varepsilon^2} < n$ whenever $n \geq N$. Choose $N \geq \frac{1}{\varepsilon^2}$. Then, $\forall n \geq N,$ we have: $\left|\sqrt{n+1} - \sqrt{n}\right| < \frac{1}{\sqrt{n}} < \varepsilon.$ That is, $a_n \to 0$ as $n \to \infty$. Q.E.D.