2.2.5 Prove Theorem 2.2.6.

Theorem 2.2.6: Let $\{a_n\}$ and $\{b_n\}$ be sequences and suppose that $a_n \to a$ and $b_n \to b$. Suppose that $b \neq 0$ and $b_n \neq 0\ \forall n$. Then, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$.

Proof of theorem 2.2.6:

We know that $a_n \xrightarrow[n\to\infty]{} a$ and $b_n \xrightarrow[n\to\infty]{} b$. That is, $\forall \varepsilon > 0$, $\exists\ N_1(\varepsilon) \in \mathbf{N}$ and $\exists\ N_2(\varepsilon) \in \mathbf{N} \ni \forall n \geq \max\{N_1, N_2\}$, $|a_n - a| < \frac{\varepsilon|b|}{4}$ and $|b_n - b| < \min\left\{\frac{\varepsilon|b^2|}{4|a|}, \frac{|b|}{2}\right\}$,

Then, $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - ab_n}{bb_n}\right| = \left|\frac{a_n b - ab + ab - ab_n}{bb_n - bb + bb}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{b^2 - b(b - b_n)}\right|$.

From exercise 1.1.10 and proposition 1.1.2.b, we have that $\left|b^2 - b(b - b_n)\right|$ $\geq \left|b^2\right| - |b||b - b_n| > \left|b^2\right| - \frac{|b||b|}{2} = \frac{\left|b^2\right|}{2}$ $\forall n \geq N$ since $|b_n - b| < \min\left\{\frac{\varepsilon|b^2|}{4|a|}, \frac{|b|}{2}\right\}$,

Taking $\varepsilon > 0$ arbitrary, we have:

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{b^2 - b(b - b_n)}\right| < \frac{|b(a_n - a) - a(b_n - b)|}{\frac{\left|b^2\right|}{2}} = 2\left|\frac{b(a_n - a) - a(b_n - b)}{b^2}\right|$$

$\leq 2\left|\frac{b(a_n - a)}{b^2}\right| + 2\left|\frac{a(b_n - b)}{b^2}\right| = 2\frac{|a_n - a|}{|b|} + 2\frac{|a||b_n - b|}{\left|b^2\right|}$ by the triangle inequality.

$\forall n \geq N$, since $|a_n - a| < \frac{\varepsilon|b|}{4}$ and $|b_n - b| < \min\left\{\frac{\varepsilon|b^2|}{4|a|}, \frac{|b|}{2}\right\}$, we have:

$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq 2\frac{|a_n - a|}{|b|} + 2\frac{|a||b_n - b|}{\left|b^2\right|} < 2\frac{\varepsilon|b|}{4|b|} + 2\frac{|a|\varepsilon\left|b^2\right|}{4|a|\left|b^2\right|} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. i.e $\frac{a_n}{b_n} \to \frac{a}{b}$ as $n \to \infty$. Q.E.D.