2.2.2 Find the limits of the following sequences

2.2.2.b $a_n = (\sin n)\sin\frac{1}{n}$

Lemma 2.2.2.b:

Let $c_n = \sin\frac{1}{n}$. Then $c_n \to 0$ as $n \to \infty$.

Proof of lemma 2.2.2.b:

$\forall \varepsilon > 0,$ we want to show that $\exists\, N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, \left|\sin\frac{1}{n}\right| < \varepsilon.$

Take $\varepsilon > 0$ be arbitrary. Let $N(\varepsilon) = \frac{1}{\sin^{-1}\{\varepsilon\}}$ where $\{\varepsilon\}$ is the fractional part of $\varepsilon,$ defined in problem set 2. (If $\varepsilon > 1$, then $\sin^{-1}\varepsilon$ is not defined.)

$\sin\frac{1}{n}$ is decreasing in n since if $f(x) = \sin\frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x} < 0\ \forall x < \frac{2}{\pi}$

$\Rightarrow f(n) > f(n+1)$. Then, $\forall n \geq N,$ we have that,

$\left|\sin\frac{1}{n}\right| \leq \left|\sin(\sin^{-1}\{\varepsilon\})\right| = \left|\{\varepsilon\}\right| = \{\varepsilon\} \leq \varepsilon.$ Q.E.D.

Answer to exercise 2.2.2.b:

Define $b_n = \sin n$ and $c_n = \sin\frac{1}{n}$. Then we have that $a_n = b_n c_n$.

We know that a property of the sin function is that $\forall n, \left|\sin n\right| \leq 1$.

Thus, b_n is a bounded sequence. That is, $\exists\, M > 0 \ni \forall n \in N, \left|b_n\right| \leq M$. Namely with $M = 1$.

Since $c_n \to 0$ as $n \to \infty$ and since b_n is bounded, by exercise 2.2.4 (with proof later), $a_n = b_n c_n \to 0$ as $n \to \infty$. Therefore, the limit of a_n is 0.