2.1.5 Prove that a sequence $\{a_n\}$ can have at most one limit.

Lemma 2.1.5.a:

$\forall x, y \in \mathbf{R},$ if $x > y,$ then $\exists\ \delta > 0 \ni x - \delta > y + \delta.$

Proof of lemma 2.1.5.a:

Suppose $x > y$. Then, $x - y > 0 \Rightarrow x - y > \frac{x-y}{2} > 0$. Choose $\delta = \frac{x-y}{4}$.

$\Rightarrow x - y > \frac{x-y}{4} + \frac{x-y}{4} = 2\delta \Rightarrow x - \delta > y + \delta$ Q.E.D.

Proof of exercise 2.1.5:

In order to get a contradiction, suppose $a_n \to a$ and $a_n \to a' \neq a$.

WLOG, suppose that $a < a' \Rightarrow \ \exists \delta > 0 \ni a + \delta < a' - \delta$ by lemma 2.1.5.a.

Since $a_n \to a$ and $a_n \to a' \neq a$, we have that $\forall \varepsilon > 0$, $\exists\ N_1(\varepsilon), N_2(\varepsilon) \in \mathbf{N} \ni$

$\forall n \geq N_1$, $|a_n - a| < \varepsilon$ and $\forall n \geq N_2$, $|a_n - a'| < \varepsilon$. Choose $\varepsilon = \delta$ and let

$N = \max\{N_1(\delta), N_2(\delta)\}$. Then, $\forall n \geq N$, we have:

$|a_n - a| < \delta \Rightarrow -\delta < a_n - a < \delta \Rightarrow a - \delta < a_n < a + \delta$

$|a_n - a'| < \delta \Rightarrow -\delta < a_n - a' < \delta \Rightarrow a' - \delta < a_n < a' + \delta$

Since $a + \delta < a' - \delta$, it follows that:

$a - \delta < a_n < a + \delta < a' - \delta < a_n < a' + \delta \Rightarrow a_n < a_n$.

In particular, $a_n \neq a_n$, which is a contradiction. $\rightarrow\leftarrow$ Q.E.D.