2.1.3 Prove directly that each of the following sequences converges by letting $\varepsilon > 0$ be given and finding $N(\varepsilon)$ so that (1) holds:

2.1.3.b $a_n = \frac{3n+1}{n+2}$

Proof of exercise 2.1.3.b:

Want to show $\forall \varepsilon > 0 \ \exists \ N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N, \ \left|\frac{3n+1}{n+2} - 3\right| < \varepsilon.$

$$\left|\frac{3n+1}{n+2} - 3\right| = \left|\frac{3n+1}{n+2} - \frac{3(n+2)}{n+2}\right| = \left|\frac{3n+1}{n+2} - \frac{3n+6}{n+2}\right| = \left|\frac{3n+1-3n-6}{n+2}\right| = \left|\frac{-5}{n+2}\right|$$

$= |-1|\left|\frac{5}{n+2}\right|$ by proposition 1.1.2.b

$$|-1|\left|\frac{5}{n+2}\right| = \left|\frac{5}{n+2}\right| \leq \left|\frac{5}{n}\right| = \frac{5}{n}.$$

Take $\varepsilon > 0$ arbitrary. $\frac{5}{n} < \varepsilon$ whenever $n \geq N \Leftrightarrow \frac{5}{\varepsilon} < n$ whenever $n \geq N$.

Choose $N \geq \frac{5}{\varepsilon}$. Then we have $\forall n \geq N$,

$\left|\frac{3n+1}{n+2} - 3\right| \leq \frac{5}{n} < \varepsilon$. Thus, $\frac{3n+1}{n+2} \to 3$ as $n \to \infty$. Q.E.D.