Michael Powell; 603-210-926
Math 131A; Prof. Demeter
University of California at Los Angeles
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Problem Set 3

2.1.2 Prove directly that each of the following seqeunces converges by letting $\varepsilon > 0$ be given and finding $N(\varepsilon)$ so that (1) holds.

2.1.2.d $a_n = \sqrt{\dfrac{n}{n+1}}$

Proof of exercise 2.1.2.d:

Want to show $\forall \varepsilon > 0 \ \exists \ N(\varepsilon) \in \mathbf{N} \ni \forall \ n \geq N, \quad \left| \sqrt{\dfrac{n}{n+1}} - 1 \right| < \varepsilon.$

$$\left| \sqrt{\frac{n}{n+1}} - 1 \right| = \left| \frac{\sqrt{n}}{\sqrt{n+1}} - \frac{\sqrt{n+1}}{\sqrt{n+1}} \right| = \left| \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}} \right| = \left| \left(\frac{\sqrt{n} + \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}} \right) \left(\frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}} \right) \right|$$

$$= \left| \frac{n - n - 1}{\sqrt{n}\sqrt{n+1} + \sqrt{n+1}\sqrt{n+1}} \right| = |-1| \left| \frac{1}{\sqrt{n(n+1)} + n + 1} \right| \text{ by proposition 1.1.2.b}$$

$$|-1| \left| \frac{1}{\sqrt{n(n+1)} + n + 1} \right| \leq \left| \frac{1}{n+1} \right| \text{ since } \sqrt{n(n+1)} > 0$$

$$\left| \frac{1}{n+1} \right| \leq \left| \frac{1}{n} \right| = \frac{1}{n}.$$

Take $\varepsilon > 0$ arbitrary. $\dfrac{1}{n} < \varepsilon$ whenever $n \geq N \Leftrightarrow \dfrac{1}{\varepsilon} < n$ whenever $n \geq N$.

Choose $N \geq \dfrac{1}{\varepsilon}$. Then we have $\forall n \geq N$,

$\left| \sqrt{\dfrac{n}{n+1}} - 1 \right| \leq \dfrac{1}{n} < \varepsilon.$ Thus, $\sqrt{\dfrac{n}{n+1}} \to 1$ as $n \to \infty$. Q.E.D.