1.4.ex.1 For each $x > 0$, denote by $\{x\}$ the fractional part of x. (That is, the real number between 0 and 1 having the same decimal expansion as *x*.) For example, {5.145}=0.145. Let *a* now be an arbitrary positive real number. Show that at least one of the numbers $\{a\},\{2a\},\ldots,\{10a\}$ belongs to the interval $[0,0.1]$. (Hint: Proof by contradiction; divide the interval [0,1) into 10 intervals of length 0.1)

Proof of exercise 1.4.ex.1: (Proof by contradiction)

Divide the interval [0,1) into ten intervals of length 0.1: $I_j = \left[\frac{j-1}{10}, \frac{j}{10}\right),$ $j = 1,\ldots,10.$ Suppose $\forall k = 1,2,\ldots,10,$ we have that $\{ka\} \notin I_1$. Then, since there are ten points in question and only nine intervals in which they could be, at least one interval contains two points: $\{k_1 a\}$ and $\{k_2 a\}$ where $k_1, k_2 \in \{1,2,\ldots 10\}$ and with $k_1 > k_2$. Since both these points are contained within the same interval of length 0.1, the distance between them is at most 0.1. (The set I_j is a bounded set with diameter 0.1. Therefore, any two points within it must have distance less than 0.1) Therefore, $\{k_1 a\} - \{k_2 a\} < 0.1$. But $\{k_1 a\} - \{k_2 a\} = \{k_1 a - k_2 a\}$ $= \{(k_1 - k_2)a\}$. Since $k_1, k_2 \in \{1,2,\ldots,10\},$ it follows that $k_1 - k_2 \equiv k_3 \in \{1,2,\ldots,10\}$. Thus, $\{k_3 a\} < 0.1$. That is, $\{k_3 a\} \in [0,0.1),$ which is a contradiction. Therefore, there is at least one point in $[0,0.1) \subset [0,0.1]$. Q.E.D.