1.4.6 Prove that if n is a positive integer, then $n^3 + 5n$ is divisible by 6.

Definition: An integer k is <u>divisible by 6</u> if $\frac{k}{6} = c$ for some $c \in \mathbf{Z}$.

Definition: An integer k is <u>divisible by 2</u> if $\frac{k}{2} = c$ for some $c \in \mathbf{Z}$.

Lemma 1.4.6.1: If n is a positive integer, then $n^2 + n$ is divisible by 2.

Proof of lemma 1.4.6.1:

Since we are dealing only with positive integers, we may use mathematical induction. (Standard induction techniques will not work on the set of all integers.)

(n=1): $\frac{1^2+1}{2} = \frac{2}{2} = 1 \in \mathbf{Z}$. Thus, (n=1) holds.

Assume (n=n): $\frac{n^2+n}{2} = c \in \mathbf{Z}$.

(n=n+1): $\frac{(n+1)^2+n+1}{2} = \frac{n^2+2n+1+n+1}{2} = \frac{n^2+n}{2} + \frac{2n+2}{2}$

$\frac{n^2+n}{2} + \frac{2n+2}{2} = c + \frac{2n+2}{2}$ by induction assumption.

$\frac{(n+1)^2+n+1}{2} = (c+n+1) \equiv c' \in \mathbf{Z}$. Thus, (n=n+1) holds.

Proof of exercise 1.4.6:

Since we are dealing only with positive integers, we may use mathematical induction. (Standard induction techniques will not work on the set of all integers.)

(n=1): $\frac{1^3+5(1)}{6} = \frac{6}{6} = 1 \in \mathbf{Z}$.

Assume (n=n): $\frac{n^3+5n}{6} = c \in \mathbf{Z}$.

(n=n+1): $\frac{(n+1)^3+5(n+1)}{6} = \frac{(n+1)(n^2+2n+1)+5n+5}{6}$

$= \frac{n^3+2n^2+n+n^2+2n+1+5n+5}{6} = \frac{n^3+5n}{6} + \frac{3n^2+3n}{6} + \frac{6}{6}$

$= c + \frac{n^2+n}{2} + 1 = c + c' + 1 \equiv c'' \in \mathbf{Z}$, which follows from the induction assumption and by lemma 1.4.6.1 and the Peano axioms in that the sum of natural numbers is still a natural number. Q.E.D.