1.4.12 Prove that e is irrational by supposing that $e = \frac{m}{n}$ and deriving a contradiction.

Use the fact that $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. Let s_k be the partial sum $s_k = \sum_{j=0}^{k} \frac{1}{j!}$.

1.4.12.a Prove that $e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \dots\right\}$

Proof of 1.4.12.a:

Since $e = \sum_{j=0}^{\infty} \frac{1}{j!}$ and $s_k = \sum_{j=0}^{k} \frac{1}{j!}$, we have that:

$$e - s_k = \sum_{j=0}^{\infty} \frac{1}{j!} - \sum_{j=0}^{k} \frac{1}{j!} = \frac{1}{1!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{2!} + \dots + \frac{1}{k!} - \frac{1}{k!} + \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \dots$$

$$= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \frac{1}{(k+3)!} + \dots$$

$$= \frac{1}{(k+1)!} + \frac{1}{(k+2)(k+1)!} + \frac{1}{(k+3)(k+2)(k+1)!} + \dots$$

$$= \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+2} + \frac{1}{(k+3)(k+2)} + \dots\right\}$$

We know that:

$$k+2 > k+1 \Rightarrow \frac{1}{k+2} < \frac{1}{k+1}, \dots (k+j)(k+j-1)\dots(k+2) > (k+1)^{j-1}$$

$$\Rightarrow \frac{1}{(k+j)(k+j-1)\dots(k+2)} < \frac{1}{(k+1)^{j-1}} = \left(\frac{1}{k+1}\right)^{j-1}.$$

Thus, we have the result:

$e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+2}\right) + \dots\right\}$ as desired. Q.E.D.

1.4.12.b Prove that $e - s_k < \frac{1}{k(k!)}$ for all positive integers k.

Proof of 1.4.12.a:

By example 1 in section 6.2, we have that, if $\alpha < 1$, then $\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}$.

From part a, we know: $e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \ldots\right\}$. Since k is a positive integer, $k > 0 \Rightarrow k+1 > 1 \Rightarrow \frac{1}{k+1} < 1$. Therefore,

$$\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \ldots\right\} = \frac{1}{1 - \frac{1}{k+1}} = \frac{1}{\frac{k+1-1}{k+1}} = \frac{1}{\frac{k}{k+1}} = \frac{k+1}{k}.$$

Thus, $e - s_k < \frac{1}{(k+1)!}\left\{1 + \frac{1}{k+1} + \left(\frac{1}{k+1}\right)^2 + \ldots\right\} = \frac{1}{(k+1)!}\frac{k+1}{k}$

$= \frac{1}{(k+1)k!}\frac{k+1}{k} = \frac{1}{k(k!)}$. Q.E.D.

1.4.12.c If $e = \frac{m}{n}$, prove that $n!e$ and $n!s_n$ are integers.

Let $e = \frac{m}{n}$. Then, $n!e = \frac{n!m}{n} = (n-1)(n-2)\ldots(2)(1)m$, which is an integer. (see the construction of the integers in Tao 131AH notes, week 1).
Since $s_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$, we have that $n!s_n = \frac{n!}{1!} + \frac{n!}{2!} + \ldots + \frac{n!}{(n-1)!} + \frac{n!}{n!}$
$= n(n-1)\ldots(3)(2) + (n)(n-1)\ldots(4)(3) + \ldots n + 1$. Since $n,\ n-1,\ \ldots,\ 2, 1$ are all integers, it follows (see the construction of the integers in Tao 131AH notes, week 1) that s_n, the sum of products of integers, is an integer. Q.E.D.

1.4.12.d If $e = \frac{m}{n}$, prove that $n!(e - s_n)$ is an integer between 0 and 1, which is absurd.

Proof of exercise 1.4.12.d:

Since, from part c, we have that $n!e$ and $n!s_n$ are integers, it follows that $n!e - n!s_n$ is an integer. (See the construction of the integers in Tao 131AH notes, week 1). By axiom P9, we have that $n!e - n!s_n = n!(e - s_n)$. Pick n ≥ 2. Then, using part b, we have:

$n!(e - s_n) < n!\frac{1}{n(n!)} = \frac{1}{n}$, but since $n \geq 2$, $\frac{1}{n} \in (0,1)$. Thus, $n!(e - s_n) \in (0,1)$, which is a contradiction, since $n!(e - s_n)$ is an integer. Therefore, e must be irrational. Q.E.D.