1.3.ex.1 Show that there exists an injective function f defined on $\boldsymbol{R}$ and with values in $P(Q)$. Deduce that $\boldsymbol{R}$ has the same cardinality with a subset of $P(Q)$.

Proof of exercise 1.3.ex.1:

For all $x \in \mathbf{R},$ there is a "decimal expanision" of x: $x = x_0.x_1x_2x_3...$

Define the function $f : \mathbf{R} \to P(\mathbf{Q})$ by

$f(x) = \{x_0, x_0.x_1, x_0.x_1x_2, x_0.x_1x_2x_3,...\} \in P(\mathbf{Q})$ That is, $f(x)$ is the Cauchy sequence of rational numbers which converges to x.

I will now prove that f is injective:

Suppose $f(x) = f(x')$: $\{x_0, x_0.x_1, x_0.x_1x_2, x_0.x_1x_2x_3,...\}$ $= \{x_0', x_0'.x_1', x_0'.x_1'x_2', x_0'.x_1'x_2'x_3',...\}$. Since, by construction, for each $n \in \mathbf{N}$, there is exactly one element of $f(x)$ and one element of $f(x')$ containing exactly n digits, it follows that in order for these two sets to be equal, each element with n digits must be equal. That is, we must have: $x_0.x_1...x_n = x_0'.x_1'...x_n'$. Without going into too much detail about what it means for two rational numbers to be equal (see Tao's 131AH week 1 notes), it follows that $x_0 = x_0'$; $x_1 = x_1'$; ...; $x_n = x_n'$. Since this holds for all n it follows that $f(x)$ and $f(x')$ represent the same Cauchy sequence of rational numbers. Thus, they must have the same limit (which will not be proven here). Therefore, $x = x'$.

Since $D(f) = \mathbf{R}$ and f is injective, it follows that $\mathbf{R}$ has the same cardinality as $R(f) \subset P(\mathbf{Q})$. Q.E.D.