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Problem Set #2

1.3.9 For each of the following sets, say whether it is finite, countable, or uncountable:

1.3.9.a The set of functions from a finite set to a finite set.
Answer to exercise 1.3.9.a:

Let A, and B be finite sets. For ease of notation, enumerate their contents as: $A = \{1,2,...,n\}$, $B = \{1,2,...,m\}$.
Let $F = \{ f \mid f : \{1,2,...,n\} \to \{1,2,...,m\}\}$
Without loss of generality, assume that $D(f) = A$. To see that this is a reasonable assumption, consider the case where $D(f) = A \setminus \{j\}$. Denote $A' = \{1,2,..., j-1, j+1,..,n\}$. The same analysis that follows will still apply, except that the domain will contain one fewer element.
For each $a \in A$, there are m different elements to which it can be mapped. Since there are n such elements in A, it follows that there are a total of m^n different combinations of possible mappings. Thus, $\#F = m^n$, which is finite.

1.3.9.b The set of functions from a finite set to a countable set.
Lemma 1.3.9.b.1:

$\mathbf{N} \times \mathbf{N} \times ... \times \mathbf{N}$ (n times) is countable.

Proof of lemma 1.3.9.b.1:

Induct on n.
$(n = 1)$: $\mathbf{N}$ is countable. This is trivially true as the identity map from $\mathbf{N} \to \mathbf{N}$ is bijective. Thus, $(n = 1)$ holds.
Assume $(n = n)$: $\mathbf{N} \times ... \times \mathbf{N}$ (n times) is countable.
$(n = n+1)$: $\mathbf{N} \times ... \times \mathbf{N} \times \mathbf{N}$ (n+1 times) = $(\mathbf{N} \times ... \times \mathbf{N}) \times \mathbf{N}$. By proposition 1.3.4, since $\mathbf{N} \times ... \times \mathbf{N}$ (n times) is countable and $\mathbf{N}$ is countable, $\mathbf{N} \times ... \times \mathbf{N} \times \mathbf{N}$ (n+1 times) is countable. Thus, $(n = n+1)$ holds. Q.E.D.

Answer to exercise 1.3.9.b:

Let $F = \{ f \mid f : A \to B\}$ where A is finite and B is countable. For ease of notation, let $A = \{1, ..., n\}$, and $B = \{1,2,3, ...\}$.
Define $g : F \to \mathbf{N} \times ... \times \mathbf{N}$ (n times) by:
$g(f) = (f(1), ..., f(n))$.
Suppose $g(f) = g(f')$. Then, $(f(1), ..., f(n)) = (f'(1), ..., f'(n))$. That is, $f = f'$. Therefore, g is injective.
Take $k \in \mathbf{N} \times ... \times \mathbf{N} \ni k = (k_1, ..., k_n)$.
Define $f \in F$ by $f(1) = k_1, ..., f(n) = k_n$. Therefore, $\forall k \in \mathbf{N} \times ... \times \mathbf{N}$, $\exists f \in F \ni g(f) = k$. That is, g is surjective.

$\Rightarrow$ F and $\mathbf{N}\times\ldots\times\mathbf{N}$ have the same cardinality. By Lemma 1.3.9.b.1, F is countable.

1.3.9.c The set of functions from a countable set to a finite set with two or more elements.

Lemma 1.3.9.c:

$\mathbf{N}\times\mathbf{N}\times\ldots$ is uncountable.

Proof of lemma 1.3.9.c: (by contradiction)

Suppose $\mathbf{N}\times\mathbf{N}\times\ldots$ is countable. Then $\exists\ f:\mathbf{N}\to\mathbf{N}\times\mathbf{N}\times\ldots$ bijective.

Take $n^1\in\mathbf{N}\times\mathbf{N}\times\ldots,\ n^2\in\mathbf{N}\times\mathbf{N}\times\ldots,\ n^3\in\mathbf{N}\times\mathbf{N}\times\ldots,\ \ldots$ such that:

$$f(1)=(n_1^{\ 1},n_2^{\ 1},n_3^{\ 1},\ldots);$$
$$f(2)=(n_1^{\ 2},n_2^{\ 2},n_3^{\ 2},\ldots);$$
$$f(3)=(n_1^{\ 3},n_2^{\ 3},n_3^{\ 3},\ldots);$$
$$\vdots$$

Construct $n^*=(n_1^*,n_2^*,n_3^*,\ldots)$ such that $n_1^*\neq n_1^{\ 1},\ n_2^*\neq n_2^{\ 2},\ n_3^*\neq n_3^{\ 3},\ldots$

Clearly, $\exists\ n\in\mathbf{N}\times\mathbf{N}\times\ldots\ni f(j)\neq n^*\ \forall j\in\mathbf{N}$. That is, f is not surjective, which is a contradiction. Thus, $\mathbf{N}\times\mathbf{N}\times\ldots$ is uncountable.

Answer to exercise 1.3.9.c:

Let $F=\{f\mid f:A\to B\}$ where A is countable and B is finite. For ease of notation, we can denote $A=\{a_1,a_2,\ldots\}$ and $B=\{b_1,\ldots,b_n\}$.

Define $g:F\to\mathbf{N}\times\mathbf{N}\times\ldots$ by $g(f)=(f(a_1),f(a_2),\ldots)$.

Suppose $g(f)=g(f')$. Then $(f(a_1),f(a_2),\ldots)=(f'(a_1),f'(a_2),\ldots)$. That is, $f(a_1)=f'(a_1),\ f(a_2)=f'(a_2),\ \ldots$ i.e. $f=f'$. Thus, g is injective.

Take $(n_1,n_2,\ldots,)\in\mathbf{N}\times\mathbf{N}\times\ldots$ Define $f\in F$ by $f(a_1)=n_1,\ f(a_2)=n_2,\ldots$ Thus, g is surjective.

Therefore, g is bijective and $D(g)=F$. $\Rightarrow$ F has the same cardinality as $\mathbf{N}\times\mathbf{N}\times\ldots$. By lemma 1.3.9.c.1, it follows that F is uncountable.

1.3.9.d The set of all finite subsets of the integers. Hint: prove that for each *n*, the set of finite subsets of size *n* is countable.

Definition: A set is <u>at most countable</u> if it is either finite or countable.

Definition: Two sets A and B are <u>equal</u> iff $x\in A\Rightarrow x\in B$ and $x\in B\Rightarrow x\in A$.

Lemma 1.3.9.d.1:

$B_n=\{A\subset\mathbf{Z}\mid \#A=n\}$ is at most countable.

Proof of lemma 1.3.9.d.1:

Proof by induction.

$(n=0)$: This is trivially true as the only subset with zero elements is the empty set, which is finite by definition.

$(n=1)$: This, too, is trivially true. Each element $A^j \in B_1$ consists of a singleton $\{j\}$. Define $f: B_n \to \mathbf{Z}$ by $f(\{j\}) = j$. Thus, f is a bijection between B_1 and $\mathbf{Z}$, which is a countable set. Hence, B_1 is countable.

Assume $(n=n)$: B_n is countable.

$(n=n+1)$: Want to show that $B_{n+1} = \{A \subset \mathbf{Z} \mid \# A = n+1\}$ is countable.

$B_1 \times B_n$ is countable by proposition 1.3.4 since B_1 and B_n are countable.

Take $A \in B_{n+1}$. Since $\# A = n+1$, A is finite. Define $f: \{1,..., n+1\} \to A$ by $f(1) = \max_a \{a \in A\} \equiv a_1$, $f(2) = \max_a \{a \in A \setminus \{a_1\}\} \equiv a_2$,…, $f(n+1) = \max_a \{a \in A \setminus \{a_1,..., a_n\}\} \equiv a_{n+1}$. Clearly, f is bijective with $D(f) = \{1,..., n+1\}$. Thus, $\forall A \in B_{n+1}$, we can "order" the elements from greatest to least and denote $A = \{a_1,..., a_{n+1}\}$.

Define $g: B_{n+1} \to B_1 \times B_n$ by $g(A) = (\{a_1\}, \{a_2,..., a_{n+1}\})$.

Clearly, $D(g) = B_{n+1}$ and $R(g) \subset B_1 \times B_n$. Therefore, B_{n+1} is countable if we can show that g is injective by proposition 1.3.2.

Suppose $g(A) = g(A')$. Then $(\{a_1\}, \{a_2,..., a_{n+1}\}) = (\{a_1'\}, \{a_2',..., a_{n+1}'\})$. Thus, $\{a_1\} = \{a_1'\}$ and $\{a_2,..., a_{n+1}\} = \{a_2',..., a_{n+1}'\}$. By definition, two sets are equal iff each element of one set is an element of the other. Thus, $a_1 = a_1'$ and $a_j \in \{a_2,..., a_{n+1}\} \Leftrightarrow a_j \in \{a_2',..., a_{n+1}'\}$. That is, the *j*th largest element of A is also the *j*th largest element of A' $\forall j = 2,..., n+1$. Therefore, $a_2 = a_2'$,..., $a_{n+1} = a_{n+1}'$. Thus, $\{a_1,..., a_{n+1}\} = \{a_1',..., a_{n+1}'\}$. i.e. $A = A'$.

Therefore, $(n=n+1)$ is true.

Answer to 1.3.9.d:

It was established in discussion section that the countable union of at most countable sets is countable. Thus, $\bigcup_{n=0}^{\infty} \{A \subset \mathbf{Z} \mid \# A = n\}$ is countable by lemma 1.3.9.d.1 since $B_n = \{A \subset \mathbf{Z} \mid \# A = n\}$ is finite for $n = 0$ and countable for $n \geq 1$.