1.3.5 Prove that the set of two-by-two matrices with rational entries is countable.
Proof of exercise 1.3.5:

Let $A = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbf{Q} \right\}$,

$B = \{(a,b,c,d) \mid a,b,c,d \in \mathbf{Q}\}$,

$C = \{(a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2) \mid a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2 \in \mathbf{Z}\}$,

Define $f : A \to B$ by:

$$f\left(\begin{bmatrix} a^h & b^h \\ c^h & d^h \end{bmatrix} \right) = (a^h, b^h, c^h, d^h)$$

Clearly, f is a bijection and $D(f) = A$. Thus, $Card(A) = Card(B)$

Since every component of every element of B is a rational number, it can be expressed as the ratio of two integers:

$$a = \frac{a_1}{a_2}; \qquad b = \frac{b_1}{b_2}; \qquad c = \frac{c_1}{c_2}; \qquad d = \frac{d_1}{d_2};$$

$a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2 \in \mathbf{Z}$ and

$a_2 \neq 0,\ b_2 \neq 0,\ c_2 \neq 0,\ d_2 \neq 0.$

Thus, $B = \left\{ \left(\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}, \frac{d_1}{d_2} \right) \mid \begin{matrix} a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2 \in \mathbf{Z}, \\ a_2 \neq 0, b_2 \neq 0, c_2 \neq 0, d_2 \neq 0. \end{matrix} \right\}$

Define $g : B \to C$ by:

$$\mathrm{f}\left(\frac{a_1{}^h}{a_2{}^h}, \frac{b_1{}^h}{b_2{}^h}, \frac{c_1{}^h}{c_2{}^h}, \frac{d_1{}^h}{d_2{}^h} \right) = (a_1{}^h, a_2{}^h, b_1{}^h, b_2{}^h, c_1{}^h, c_2{}^h, d_1{}^h, d_2{}^h)$$

Clearly, g is a bijection and $D(g) = B$. Thus, $Card(B) = Card(C)$. By Proposition 1.3.1, $Card(A) = Card(C)$.

Define $h : C \to D \subset \mathbf{N}$ by:

$$g(a_1{}^h, a_2{}^h, b_1{}^h, b_2{}^h, c_1{}^h, c_2{}^h, d_1{}^h, d_2{}^h) = 2^{h_1}3^{h_2}5^{h_3}7^{h_4}11^{h_5}13^{h_6}17^{h_7}23^{h_8}, \text{ where:}$$

$$h_1 = \begin{cases} 2a_1{}^h, a_1{}^h \geq 1 \\ 1 - 2a_1{}^h, a_1{}^h \leq 0 \end{cases};\ h_2 = \begin{cases} 2a_2{}^h, a_2{}^h \geq 1 \\ 1 - 2a_2{}^h, a_2{}^h \leq 0 \end{cases};\ h_3 = \begin{cases} 2b_1{}^h, b_1{}^h \geq 1 \\ 1 - 2b_1{}^h, b_1{}^h \leq 0 \end{cases};$$

$$h_4 = \begin{cases} 2b_2{}^h, b_2{}^h \geq 1 \\ 1 - 2b_2{}^h, b_2{}^h \leq 0 \end{cases};\ h_5 = \begin{cases} 2c_1{}^h, c_1{}^h \geq 1 \\ 1 - 2c_1{}^h, c_1{}^h \leq 0 \end{cases};\ h_6 = \begin{cases} 2c_2{}^h, c_2{}^h \geq 1 \\ 1 - 2c_2{}^h, c_2{}^h \leq 0 \end{cases};$$

$$h_7 = \begin{cases} 2d_1{}^h, d_1{}^h \geq 1 \\ 1 - 2d_1{}^h, d_1{}^h \leq 0 \end{cases};\ h_8 = \begin{cases} 2d_2{}^h, d_2{}^h \geq 1 \\ 1 - 2d_2{}^h, d_2{}^h \leq 0 \end{cases}.$$

By The Fundamental Theorem of Arithmetic, h is one-to-one. h maps each element of C into an infinite subset $D \subset \mathbf{N}$. Thus, by Proposition 1.3.2, D is countable. Thus, $Card(D) = Card(\mathbf{N})$. Therefore, by Proposition 1.3.1, we have: $Card(A) = Card(\mathbf{N})$, i.e. A is countable. Q.E.D.