Definition: Let A be a set whose elements have been put in a one-to-one correspondence with {1,…,n} if A is finite and **N** if A is infinite. Let B be a set of elements of A with some property $p(B)$. The <u>first element of B</u> is defined to be $\arg\min_{j\in\{1,...,n\}}\{j \mid a_j \in A, p(B) holds\}$ if B is finite and $\arg\min_{j\in\mathbf{N}}\{j \mid a_j \in A, p(B) holds\}$ if B is infinite.

1.3.1.a Prove that the union of two finite sets is finite.
Proof of exercise 1.3.1.a:

Let A, B be finite sets. Then for some $n \in \mathbf{N}\ \exists\ f:\{1,...,n\}\to A$ bijective defined by $f(j)=a_j\ \ \forall\ j\in\{1,...,n\}$. Thus, we can write $A=\{a_1,a_2,...,a_n\}$.

Since B is finite, for some $m \in \mathbf{N}\ \exists\ g:\{1,...,m\}\to B$ bijective defined by $g(l)=b_l\ \forall\ l\in\{1,...,m\}$. Thus, we can write $B=\{b_1,b_2,...,b_m\}$.

Let $m_1 \geq 1$ correspond to the first element $b_{m_1} \in B \ni b_{m_1} \notin A.$

Let $m_2 > m_1$ correspond to the first element $b_{m_2} \in B\setminus\{b_{m_1}\} \ni b_{m_2} \notin A.$

$\vdots$

Let $m_l > m_{l-1}$ correspond to the first element $b_{m_l} \in B\setminus\{b_{m_1},...,b_{m_{l-1}}\} \ni b_{m_l} \notin A$ with $B\setminus A=\{b_{m_1},...,b_{m_l}\}$.

It follows that $A\bigcup B=\{a_1,...,a_n,b_{m_1},...,b_{m_l}\}$. Define $h: A\bigcup B\to\{1,...,n,n+1,...,n+l\}$ by:

$$
\begin{aligned}
a_1 &\mapsto 1\\
&\vdots\\
a_n &\mapsto n\\
b_{m_1} &\mapsto n+1\\
&\vdots\\
b_{m_l} &\mapsto n+l
\end{aligned}
$$

Clearly, $h: A\bigcup B\to\{1,...,n,n+1,...,n+l\}$ is bijective and $D(h)=A\bigcup B$.
Thus, $A\bigcup B$ is finite and $\text{Card}(A\bigcup B)=n+l$.

1.3.1.b Prove that the union of a finite set and a countable set is countable.
Proof of exercise 1.3.1.b:

Let A be a finite set. Then for some $n \in \mathbf{N}\ \exists\ f:\{1,...,n\}\to A$ bijective defined by $f(j)=a_j\ \ \forall\ j\in\{1,...,n\}$. Thus, we can write $A=\{a_1,a_2,...,a_n\}$.

Let B be a countable set. Then $\exists\ g:\mathbf{N}\to B$ defined by $g(l)=b_l\ \forall\ l\in\mathbf{N}$. Thus, we can write $B=\{b_1,b_2,...,b_m,...\}$.

Let $m_1 \geq 1$ correspond to the first element $b_{m_1} \in B \ni b_{m_1} \notin A.$

Let $m_2 > m_1$ correspond to the first element $b_{m_2} \in B\setminus\{b_{m_1}\} \ni b_{m_2} \notin A.$

$\vdots$

Let $m_l > m_{l-1}$ correspond to the first element $b_{m_l} \in B\setminus\{b_{m_1},...,b_{m_{l-1}}\} \ni b_{m_l} \notin A$

$\vdots$

with $B \setminus A = \{b_{m_1},...,b_{m_l},...\}$.

It follows that $A \bigcup B = \{a_1,...,a_n, b_{m_1},...,b_{m_l}\}$. Define

Define $h : A \bigcup B \to \mathbf{N}$ by:

$a_1 \mapsto 1$

$\vdots$

$a_n \mapsto n$

$b_{m_1} \mapsto n+1$

$\vdots$

$b_{m_l} \mapsto n+l$

$\vdots$

Clearly $h : A \bigcup B \to \mathbf{N}$ is bijective and $D(h) = A \bigcup B$.

Thus, $A \bigcup B$ is countable.

1.3.1.c Prove that the union of two countable sets is countable.

Proof of exercise 1.3.1.c:

Let A be a countable set. Then $\exists\ f : \mathbf{N} \to A$ defined by $f(k) = a_k\ \forall\ k \in \mathbf{N}$.

Thus, we can write $A = \{a_1, a_2,...,a_n,...\}$.

Let B be a countable set. Then $\exists\ g : \mathbf{N} \to B$ defined by $g(l) = b_l\ \forall\ l \in \mathbf{N}$. Thus, we can write $B = \{b_1, b_2,...,b_m,...\}$.

There are two cases:

Case I: $B \setminus A$ is finite. $\Rightarrow A \bigcup (A \setminus B) = A \bigcup B$ is countable by part b.

Case II: $B \setminus A$ is infinite.

Let $m_1 \geq 1$ correspond to the first element $b_{m_1} \in B \ni b_{m_1} \notin A$.

Let $m_2 > m_1$ correspond to the first element $b_{m_2} \in B \setminus \{b_{m_1}\} \ni b_{m_2} \notin A$.

$\vdots$

Let $m_l > m_{l-1}$ correspond to the first element $b_{m_l} \in B \setminus \{b_{m_1},...,b_{m_{l-1}}\} \ni b_{m_l} \notin A$

$\vdots$

with $B \setminus A = \{b_{m_1},...,b_{m_l},...\}$.

Define $f : A \bigcup B \to \mathbf{N}$ by:

$a_1 \mapsto 1$

$b_{m_1} \mapsto 2$

$a_2 \mapsto 3$

$b_{m_2} \mapsto 4$

$\vdots$

Clearly $h : A \bigcup B \to \mathbf{N}$ is bijective and $D(h) = A \bigcup B$.

Thus, $A \bigcup B$ is countable.